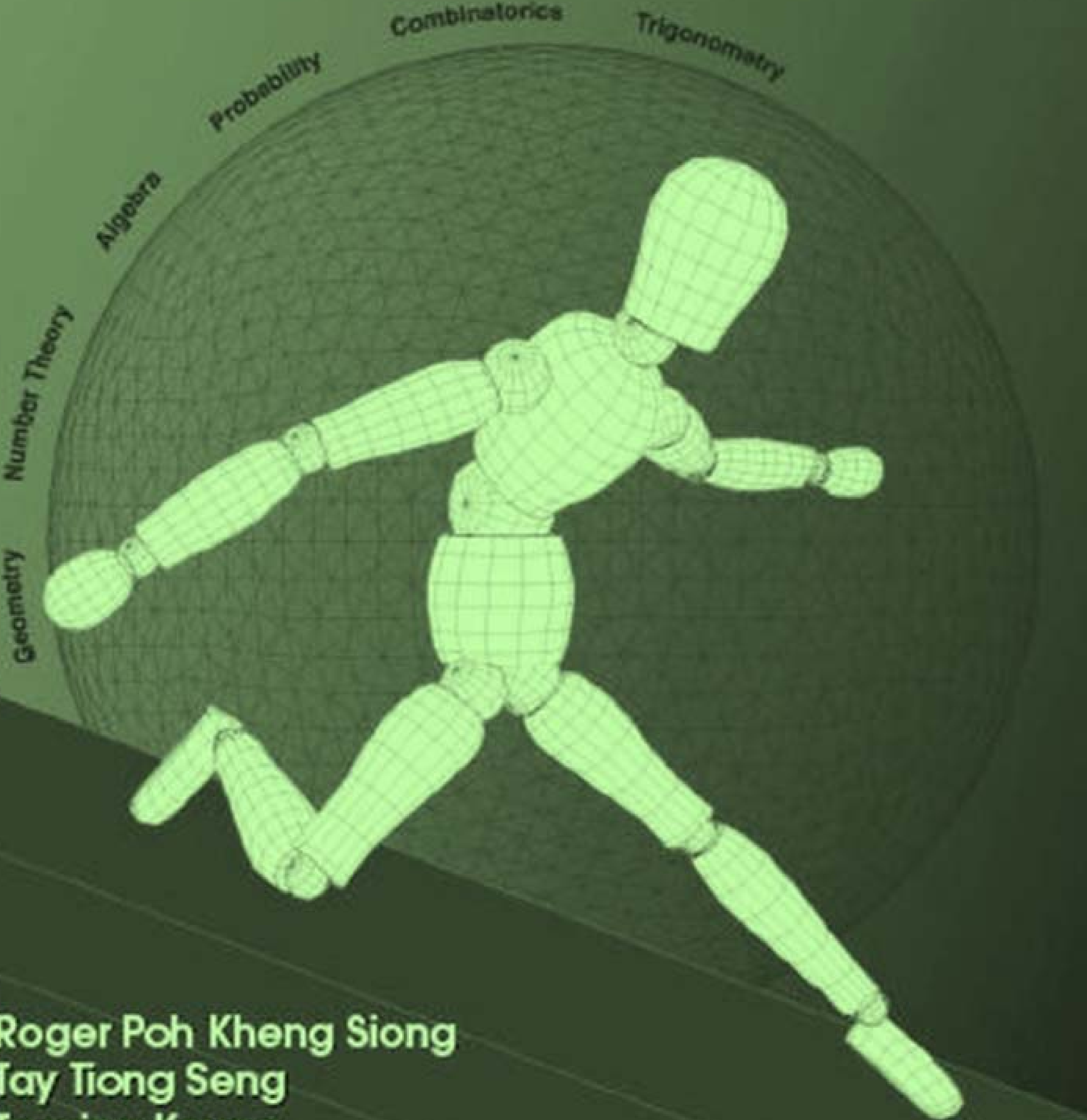
SINGAPORE MATHEMATICAL OLYMPIADS 2005



Roger Poh Kheng Siong
Tay Tiong Seng
To wing Keung
Yang Yue

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Singapore Mathematical Olympiad (SMO) 2005

(Junior Section)

Tuesday, 31 May 2005 0930-1200

Important:
Answer ALL 35 questions.
Enter your answers on the answer sheet provided.
For the multiple choice questions, enter only the letters (a, b, c, d, or e) corresponding to the correct answers in the answer sheet.
For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.
No steps are needed to justify your answers.
Each question carries 1 mark.
No calculators are allowed.

1. What is the units digit of 7^5?

   (a) 1; (b) 3; (c) 5; (d) 7; (e) 9.

2. What is the value of $(0.5)^3 + 3 \times (0.5)^2 \times (-1.5) + 3 \times (0.5) \times (-1.5)^2 + (-1.5)^3$?

   (a) 1; (b) -1; (c) 0; (d) 2; (e) -2.

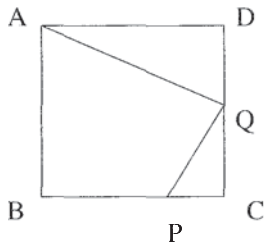
3. What is the last two digits of $1 + 2 + 3 + 4 + \cdots + 2003 + 2004 + 2005$?

   (a) 00; (b) 15; (c) 25; (d) 50; (e) 75.

4. How many zeroes does the number $50 \times 49 \times 48 \times \cdots \times 3 \times 2 \times 1$ end with?

   (a) 8; (b) 9; (c) 10; (d) 11; (e) 12.

5. In a square $ABCD$, let P be a point on the side BC such that $BP = 3PC$ and Q be the mid-point of CD. If the area of the triangle PCQ is 5, what is the area of triangle QDA?



(a) 5; (b) 10; (c) 15; (d) 20; (e) 25.

6. The values of p which satisfy the equations

$$\begin{aligned} p^2 + 9q^2 + 3p - pq &= 30 \\ p - 5q - 8 &= 0 \end{aligned}$$

may be found by solving

(a) $x^2 - x - 6 = 0$; (b) $13x^2 - 121x - 426 = 0$; (c) $29x^2 - 101x - 174 = 0$;

(d) $29x^2 + 115x - 1326 = 0$; (e) $39x^2 - 104x - 174 = 0$.

7. Using the digits $1, 2, 3, 4$ only once to form a 4-digit number, how many of them are divisible by 11?

(a) 4; (b) 5; (c) 6; (d) 7; (e) 8.

8. If a polygon has its sum of interior angles smaller than $2005°$, what is the maximum number of sides of the polygon?

(a) 11; (b) 12; (c) 13; (d) 14; (e) 15.

9. The last 2 digits of

$$2005 + 2005^2 + 2005^3 + \cdots + 2005^{2005}$$

is:

(a) 00; (b) 05; (c) 25; (d) 50; (e) 75.

10. Suppose 3 distinct numbers are chosen from $1, 2, \ldots, 3n$ with their sum equal to $3n$. What is the largest possible product of those 3 numbers?

(a) $n^3 - n$; (b) n^3; (c) $n^3 + n$; (d) $n^3 - 7n + 6$; (e) $n^3 - 7n - 6$.

11. Suppose $a \neq 0, b \neq 0$ and $\frac{b}{a} = \frac{c}{b} = 2005$. Find the value of $\frac{b+c}{a+b}$.

12. Find the exact value of $\sqrt{\frac{x-2}{6}}$ when $x = 2006^3 - 2004^3$.

13. Suppose $x - y = 1$. Find the value of

$$x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4.$$

14. Simplify

$$\frac{(1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2})\ldots(1-\frac{1}{2005^2})}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})\ldots(1-\frac{1}{2005})}.$$

15. The figure $ABCDEF$ is a regular hexagon. Evaluate the quotient

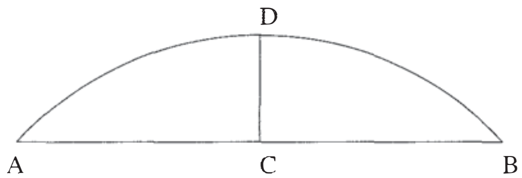
$$\frac{\text{Area of hexagon } ABCDEF}{\text{Area of triangle } ACD}.$$

16. Suppose x and y are two real numbers such that $x+y=10$ and $x^2+y^2=167$. Find the value of x^3+y^3.

17. Find the value (in the simplest form) of $\sqrt{9+4\sqrt{5}}-\sqrt{9-4\sqrt{5}}$.

18. If $x>0$ and $(x+\frac{1}{x})^2=25$, find the value of $x^3+\frac{1}{x^3}$.

19. The diagram shows a segment of a circle such that CD is perpendicular bisector of the chord AB. Given that $AB=16$ and $CD=4$, find the diameter of the circle.



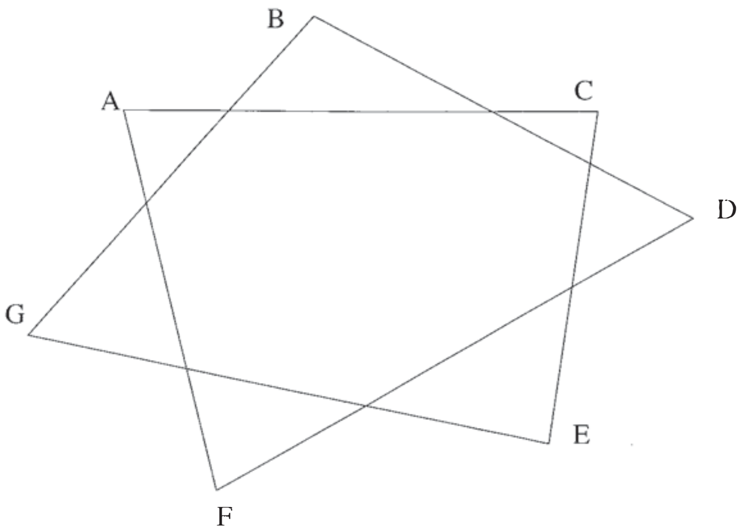
20. A palindrome number is the same either read from left to right or right to left, for example, 121 is a palindrome number. How many 5-digit palindrome numbers are there together?

21. Let p be a prime number such that the next larger number is a perfect square. Find the sum of all such prime numbers. (For example, if you think that 11 and 13 are two such prime numbers, then the sum is 24.)

22. In the figure below, if

$$\angle A+\angle B+\angle C+\angle D+\angle E+\angle F+\angle G=x \text{ degrees},$$

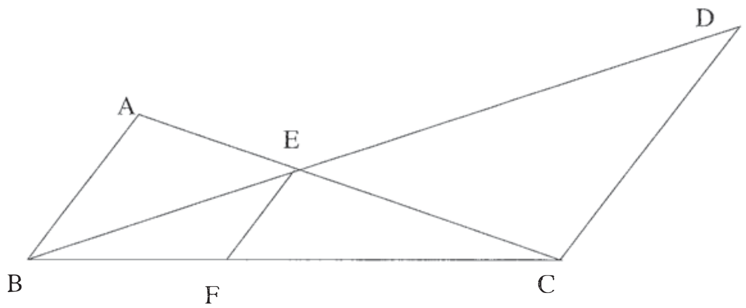
then what is x?



23. Given that $x = 2a^5 = 3b^2$ where a and b are positive integers, find the least possible value of x.

24. If $x^2 + x - 1 = 0$, find the value of $x^4 - 3x^2 + 3$.

25. In the diagram, $AB \,//\, EF \,//\, DC$. Given that $AC+BD = 250$, $BC = 100$ and $EC+ED = 150$, find CF.



26. Find the sum of all possible values of a such that the following equation has real root in x:

$$(x - a)^2 + (x^2 - 3x + 2)^2 = 0.$$

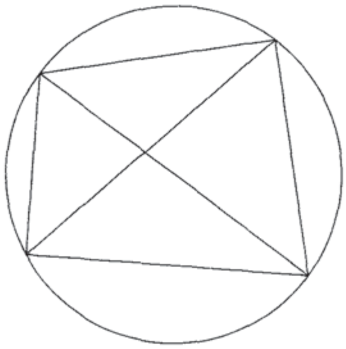
27. If two positive integers m and n, both bigger than 1, satisfy the equation

$$2005^2 + m^2 = 2004^2 + n^2,$$

find the value of $m + n$.

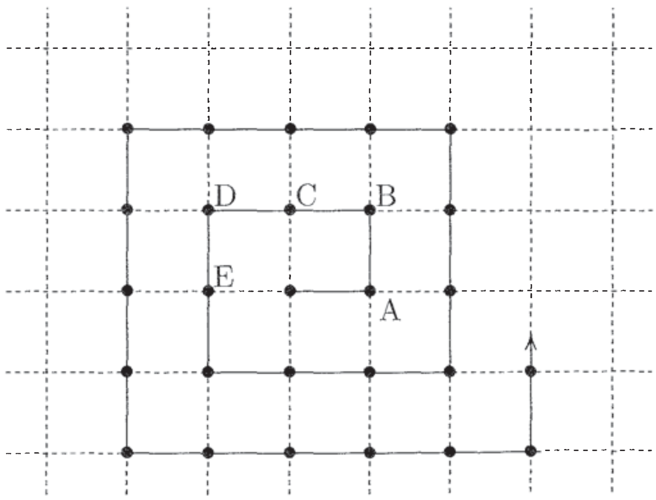
28. Suppose $a \neq 0, b \neq 0, c \neq 0$ and $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$. Find the value of $\frac{a+b-c}{a-b+c}$.

29. n dots are drawn on the circumference of a circle. By joining all the dots to one another by straight lines , the maximum number of regions that can be formed in the circle is counted. For example, when $n = 4$, the maximum number of regions is 8.

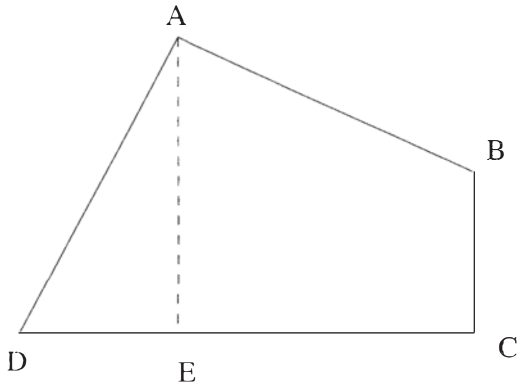


What is the maximum number of regions that can be formed when $n = 7$?

30. You walk a spiraling maze on the Cartesian plane as follows: starting at $(0, 0)$ and the first five stops are at A $(1, 0)$, B $(1, 1)$, C $(0, 1)$, D $(-1, 1)$ and E $(-1, 0)$. Your ninth stop is at the point $(2, -1)$ and so on (see the diagram below). What is the x-coordinate of the point which you would arrive at on your 2005-th stop?



31. In the following figure, $AD = AB$, $\angle DAB = \angle DCB = \angle AEC = 90°$ and $AE = 5$. Find the area of the quadrangle $ABCD$.



32. Suppose that a, b, c are distinct numbers such that

$$(b-a)^2 - 4(b-c)(c-a) = 0,$$

find the value of $\frac{b-c}{c-a}$.

33. Find the number of even digits in the product of the two 10-digit numbers

$$2222222222 \times 9999999999.$$

34. Find an integer x that satisfies the equation

$$x^5 - 101x^3 - 999x^2 + 100900 = 0.$$

35. Determine the second smallest prime factor of

$$\frac{1^3+1}{1+1} + \frac{2^3+1}{2+1} + \frac{3^3+1}{3+1} + \cdots + \frac{2005^3+1}{2005+1}.$$

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(Junior Section Solutions)

1 Ans: (d)
$7^5 = (7^4) \times 7 \equiv 7 \pmod{10}$ since $7^4 \equiv 1 \pmod{10}$.

2 Ans: (b).
Let $f(a,b) = (a+b)^3 = a^3 + 3a^2b + 3b^2 + b^3$. Then $f(0.5, -1.5) = (0.5-1)^3 = -1$.

3 Ans: (b).
$1 + 2 + \ldots + 2005 = \frac{1}{2} \times 2005 \times 2006$. Hence the last two digits is 15.

4 Ans: (e).
By counting the number of factor 5 in the product, we easily see that the number 50! ends with $\frac{50}{5} + \frac{50}{25} = 12$ zeros.

5 Ans: (d).
The two triangles are similar and $\frac{PC}{QD} = 2$. Hence Area QDA= $4\times$ Area $PCQ = 20$.

6 Ans: (a).
Substituting $q = \frac{1}{5}(p-8)$ into the first equation, we get (a).

7 Ans: (e).
If the first digit is 1, then the number is divisible by 11 implies that the 3rd digit is 4 and there are two ways to put 2 and 3. The cases when the first digit are 2, 3, 4 respectrively can be dealt with similarly. Hence the total number is $4 \times 2 = 8$.

8 Ans: (c).
The sum of interior angles is $(n-2) \times 180° < 2005°$. Hence the maximum n is 13.

9 Ans: (b).
The last 2 digits of 2005^n is 25 except $n = 1$ which gives 5. Hence the last 2 digits is $5 + 2004 \times 25 \pmod{100} = 05$.

10 Ans: (a).

From the identity $4ab = (a+b)^2 - (a-b)^2$, one sees that when $a+b$ is fixed, the product is the largest when $|a-b|$ is the smallest.

Now let $a < b < c$ be 3 distinct positive integers such that $a+b+c = 3n$ and the product abc is the biggest possible. If a and b differ more than 2, then by increasing a by 1 and decreasing b by 1, the product abc will become bigger. Hence a and b differ by at most 2. Similarly b and c differ by at most 2.

On the other hand, if $c = b+2 = a+4$, then one can increase the value of abc by decreasing c by 1 and increasing a by 1. So this case is ruled out.

Since $a+b+c$ is divisible by 3, it is not possible that $c = b+1 = a+3$ or $c = b+2 = a+3$.

So we must have $c = b+1 = a+2$. Hence the three numbers are $n-1$, n and $n+1$. Thus, their product is $(n-1)n(n+1) = n^3 - n$.

11 Ans: 2005.

$b = 2005a$ and $c = 2005b \Longrightarrow b+c = 2005(a+b)$. Hence $\dfrac{b+c}{a+b} = 2005$.

12 Ans: 2005.

Let $n = 2005$. Then $x = (n+1)^3 - (n-1)^3 = 6n^2 + 2$. Hence $\sqrt{\dfrac{x-2}{6}} = 2005$.

13 Ans: 1.

The expression $= (x-y)[(x-y)(x^2+xy+y^2) - 3xy]$. Substituting $x - y = 1$, one sees that the expression $= (x-y)^2 = 1$.

14 Ans: 1003.

The expression $= \displaystyle\prod_{i=2}^{2005}(1+\frac{1}{i}) = \prod_{i=2}^{2005}\frac{i+1}{i} = 1003$.

15 Ans: 3.

Let O be the center of the regular hexagon. Then Area OCD = Area $OAC = \dfrac{1}{6}\times$ Area $ABCDEF$. The result follows.

16 Ans: 2005.

$167 = (x+y)^2 - 2xy \Longrightarrow xy = -\dfrac{67}{2}$. Hence $x^3+y^3 = (x+y)(x^2-xy+y^2) = 2005$.

17 Ans: 4.

Completing the squares, one sees that the expression $= \sqrt{(2+\sqrt{5})^2} - \sqrt{(2-\sqrt{5})^2} = 4$.

18 Ans: 110.
As $x > 0$, we have $x + \frac{1}{x} = 5$. Thus, $x^3 + \frac{1}{x^3} = (x + \frac{1}{x})((x + \frac{1}{x})^2 - 3) = 110$.

19 Ans: 20.
Let r be the radius. Then $r^2 = (r-4)^2 + 8^2 \Longrightarrow r = 10$. Thus the diameter is 20.

20 Ans: 900.
Such a palindrome number is completely determined by its first three digits. There are $9, 10, 10$ choices of the 1st, 2nd and 3rd digits respectively, hence total $= 900$.

21 Ans: 3.
If $p + 1 = n^2$, $p = (n+1)(n-1)$. Since p is prime, $n = 2$. Hence $p = 3$ is unique.

22 Ans: 540.
By dividing up into triangles with respect to a center O, one sees that there are seven triangles formed (e.g. OAC, OBD, etc.). The middle angle at O is added up twice. Hence $x = 7 \times 180 - 720 = 540$.

23 Ans: 15552.
Observe that $3 \mid a$ and $2 \mid b$. Set $b = 2b_1$. Then we see that $2 \mid a$. Thus the smallest possible value of a is 6. Hence $x = 2(6^5) = 15552$.

24 Ans: 2.
Using long division, we get $x^4 - 3x^2 + 3 = (x^2 + x - 1)(x^2 - x - 1) + 2$.

25 Ans: 60.
From the equalities $\frac{CE}{CF} = \frac{AE}{BF} = \frac{AC}{BC}$ and $\frac{DE}{CF} = \frac{BE}{BF} = \frac{BD}{BC}$, we get

$$\frac{DE + CE}{CF} = \frac{BE + AE}{BF} = \frac{AC + BD}{BC}.$$

Thus $\frac{150}{CF} = \frac{250}{100}$, and we have $CF = 60$.

26 Ans: 3.
$x^2 - 3x + 2 = x - a = 0 \Longrightarrow a = 1$ or $a = 2$.

27 Ans: 211.
Since $(m+n)(m-n) = 4009 = 211 \times 19$ which is a product of primes. Hence $m + n = 211$.

28 Ans: 1.
Letting $\dfrac{a}{b} = k$, we get $a = bk, b = ck$ and $c = ak$. Thus $a = ak^3$. Thus $k = 1$ and $a = b = c$. The result follows.

29 Ans: 57.
One can count directly that the maximum number of regions is 57.

30 Ans: 3.
Observe that the stop numbers $1, 9, 25, 49, \ldots$ are at the lower right corners. The point $(0, -n)$ is at the stop number $(2n+1)^2 - (n+1) = 4n^2 + 3n$. When $n = 22$, we have $4(22^2) + 3(22) = 2002$. So the point $(0, -22)$ is the 2002-th stop. Thus the point $(3, -22)$ is the 2005-th stop.

31 Ans: 25.
Upon rotating AED anticlockwise 90° through A, we get a square. Thus the area is $5^2 = 25$.

32 Ans: 1.
Observe that $((b-c)+(c-a))^2 - 4(b-c)(c-a) = 0$. Hence $((b-c)-(c-a))^2 = 0$. Thus $b - c = c - a$.

33 Ans: 10.

$$\begin{aligned} 2222222222 \times (10^{10} - 1) &= 22222222220000000000 - 2222222222 \\ &= 22222222217777777778. \end{aligned}$$

34 Ans: 10.
Upon rewriting the equation as $(x^2 - 101)(x^3 - 999) + 1 = 0$, one sees that $x = 10$ is the only integral solution.

35 Ans: 11.
The sum $= \displaystyle\sum_{r=1}^{2005} (r^2 - r + 1) = \frac{1}{6} \times 2005 \times 2006 \times 4011 - \frac{1}{2} \times 2005 \times 2006 + 2005$. Upon simplifying, we see that the sum $= 5 \times 401 \times 134009 = 5 \times 11 \times 401 \times 121819\times$. Since $2, 3, 7$ does not divide 401 and 121819, the second smallest prime factor is 11.

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Each question carries 1 mark.
No calculators are allowed.

1. What is the smallest positive prime factor of the integer $2005^{2007} + 2007^{2005}$?

   (A) 5 (B) 7 (C) 2 (D) 11 (E) 3

2. What is the value of

$$\frac{2005^2 + 2 \times 2005 \times 1995 + 1995^2}{800}?$$

   (A) 20000 (B) 2000 (C) 200000 (D) 2000000 (E) None of the above

3. Let p be a real number such that the equation $2y^2 - 8y = p$ has only one solution. Then

   (A) $p < 8$ (B) $p = 8$ (C) $p > -8$ (D) $p = -8$ (E) $p < -8$

4. What is the sum of the last two digits of the integer $1! + 2! + 3! + \cdots + 2005!$?

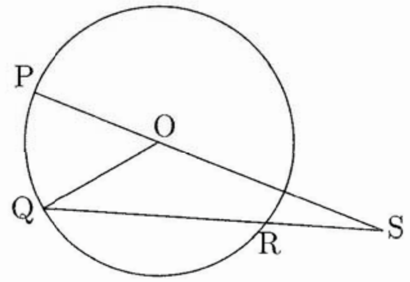
   (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

5. $5002^{2005 \log_{5002} 2005}$ is equal to

   (A) 1 (B) 2005^{2005} (C) 5002^{2005} (D) 2005^{5002} (E) 5002^{5002}

6. In the diagram, P, Q and R are three points on the circle whose centre is O. The lines PO and QR are produced to meet at S. Suppose that $RS = OP$, and $\angle PSQ = 12°$ and $\angle POQ = x°$. Find the value of x.

(A) 36 (B) 42 (C) 48 (D) 54 (E) 60



7. Let x and y be positive real numbers. What is the smallest possible value of $\dfrac{16}{x} + \dfrac{108}{y} + xy$?

(A) 16 (B) 18 (C) 24 (D) 30 (E) 36

8. In the Cartesian plane, the graph of the function $y = \dfrac{2x-1}{x-1}$ is reflected about the line $y = -x$. What is the equation for the graph of the image of the reflection?

(A) $y = \dfrac{x-1}{2x+1}$ (B) $y = \dfrac{x-1}{x+2}$ (C) $y = \dfrac{2x-1}{x+1}$ (D) $y = \dfrac{x+1}{x-2}$
(E) $y = \dfrac{-1-x}{2+x}$

9. Simplify $\sqrt{2\left(1+\sqrt{1+\left(\dfrac{x^4-1}{2x^2}\right)^2}\right)}$, where x is any positive real number.

(A) $\dfrac{x^2+1}{\sqrt{2}x}$ (B) $\dfrac{x^2+1}{x}$ (C) $\sqrt{\dfrac{x^2+1}{2x^2}}$ (D) x^2+1 (E) $\dfrac{x^2-1}{\sqrt{2}x}$

10. Let x and y be real numbers such that

$$x^2 + y^2 = 2x - 2y + 2.$$

What is the largest possible value of $x^2 + y^2$?

(A) $10 + 8\sqrt{2}$ (B) $8 + 6\sqrt{2}$ (C) $6 + 4\sqrt{2}$ (D) $4 + 2\sqrt{2}$
(E) None of the above

11. Find the greatest integer less than $(2+\sqrt{3})^4$.

12. $\triangle ABC$ is a triangle such that $\angle C = 90°$. Suppose $AC = 156$ cm, $AB = 169$ cm and the perpendicular distance from C to AB is x cm. Find the value of x.

13. Find the sum of all the real numbers x that satisfy the equation

$$(3^x - 27)^2 + (5^x - 625)^2 = (3^x + 5^x - 652)^2.$$

14. Three positive integers are such that they differ from each other by at most 6. It is also known that the product of these three integers is 2808. What is the smallest integer among them?

15. Simplify $\dfrac{2005^2(2004^2-2003)}{(2004^2-1)(2004^3+1)} \times \dfrac{2003^2(2004^2+2005)}{2004^3-1}$.

16. Consider the function $f(x) = \dfrac{1}{3^x+\sqrt{3}}$. Find the value of

$$\sqrt{3}\big[f(-5)+f(-4)+f(-3)+f(-2)+f(-1)+f(0)+f(1)+f(2)+f(3)+f(4)+f(5)+f(6)\big].$$

17. Let A and B be two positive four-digit integers such that $A \times B = 16^5 + 2^{10}$. Find the value of $A + B$.

18. A triangle $\triangle ABC$ is inscribed in a circle of radius 4 cm. Suppose that $\angle A = 60°$, $AC - AB = 4$ cm, and the area of $\triangle ABC$ is x cm^2. Find the value of x^2.

19. Let a, b and c be real numbers such that

$$a = 8 - b \quad \text{and} \quad c^2 = ab - 16.$$

Find the value of $a + c$.

20. Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ be positive integers such that

$$a_1^2 + (2a_2)^2 + (3a_3)^2 + (4a_4)^2 + (5a_5)^2 + (6a_6)^2 + (7a_7)^2 + (8a_8)^2 = 204.$$

Find the value of $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8$.

21. Find the value of the positive integer n if

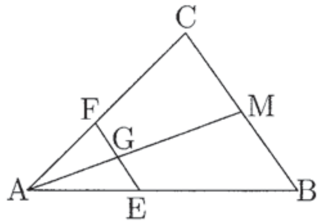
$$\frac{1}{\sqrt{4}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{6}} + \frac{1}{\sqrt{6}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{n+1}} = 10.$$

22. Let A and B be two positive prime integers such that

$$\frac{1}{A} - \frac{1}{B} = \frac{192}{2005^2 - 2004^2}.$$

Find the value of B.

23. In $\triangle ABC$, $AB : AC = 4 : 3$ and M is the midpoint of BC. E is a point on AB and F is a point on AC such that $AE : AF = 2 : 1$. It is also given that EF and AM intersect at G with $GF = 72$ cm and $GE = x$ cm. Find the value of x.



24. It is given that $x = \dfrac{1}{2-\sqrt{3}}$. Find the value of

$$x^6 - 2\sqrt{3}x^5 - x^4 + x^3 - 4x^2 + 2x - \sqrt{3}.$$

25. Let a, b and c be the lengths of the three sides of a triangle. Suppose a and b are the roots of the equation

$$x^2 + 4(c+2) = (c+4)x,$$

and the largest angle of the triangle is x°. Find the value of x.

26. Find the largest real number x such that

$$\frac{x^2 + x - 1 + |x^2 - (x-1)|}{2} = 35x - 250.$$

27. How many ways can the word MATHEMATICS be partitioned so that each part contains at least one vowel? For example, MA-THE-MATICS, MATHE-MATICS, MATHEM-A-TICS and MATHEMATICS are such partitions.

28. Consider a sequence of real numbers $\{a_n\}$ defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{a_n}{1 + na_n} \quad \text{for} \quad n \geq 1.$$

Find the value of $\dfrac{1}{a_{2005}} - 2000000$.

29. For a positive integer k, we write

$$(1+x)(1+2x)(1+3x)\cdots(1+kx) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k,$$

where $a_0, a_1, \cdots, a_k$ are the coefficients of the polynomial. Find the smallest possible value of k if $a_0 + a_1 + a_2 + \cdots + a_{k-1}$ is divisible by 2005.

30. Find the largest positive number x such that

$$(2x^3 - x^2 - x + 1)^{1+\frac{1}{2x+1}} = 1.$$

31. How many ordered pairs of integers (x, y) satisfy the equation

$$x^2 + y^2 = 2(x + y) + xy?$$

32. Find the number of ordered 7-tuples of positive integers $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ that have both of the following properties:
(i) $a_n + a_{n+1} = a_{n+2}$ for $1 \leq n \leq 5$, and
(ii) $a_6 = 2005$.

33. Let n be a positive integer such that one of the roots of the quadratic equation

$$4x^2 - (4\sqrt{3} + 4)x + \sqrt{3}n - 24 = 0$$

is an integer. Find the value of n.

34. Consider the simultaneous equations

$$\begin{cases} xy + xz = 255 \\ xz - yz = 224. \end{cases}$$

Find the number of ordered triples of positive integers (x, y, z) that satisfy the above system of equations.

35. Find the total number of positive four-digit integers N satisfying both of the following properties:
(i) N is divisible by 7, and
(ii) when the first and last digits of N are interchanged, the resulting positive integer is also divisible by 7. (Note that the resulting integer need not be a four-digit number.)

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2005

(Senior Section, Special Round)

Saturday, 9 July 2005 0930– 1230

Attempt as many questions as you can.
No calculators are allowed.
Show the steps in your calculations.
Each question carries 9 marks.

1. The digits of a 3-digit number are interchanged so that none of the digits retain their original positions. The difference of the two numbers is a 2-digit number and is a perfect square.

2. Consider the nonconvex quadrilateral $ABCD$ with $\angle C > 180^\circ$. Let the side DC extended meet AB at F and the side BC extended meet AD at E. A line intersects the interiors of the sides AB, AD, BC, CD at points K, L, J, I, respectively. Prove that if $DI = CF$ and $BJ = CE$, then $KJ = IL$.

3. Let S be a subset of $\{1, 2, 3, \ldots, 24\}$ with $|S| = 10$. Show that S has two 2-element subsets $\{x, y\}$ and $\{u, v\}$ such that $x + y = u + v$.

4. Determine whether there exists a positive integer n such that $n!$ begins with 2005. Justify your answer.

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Singapore Mathematical Olympiad 2005

(Senior Section Solutions)

1. Ans: C

   Since 2005 and 2007 are odd numbers, so are 2005^{2007} and 2007^{2005}. Thus, $2005^{2007}+2007^{2005}$ is even, and it is divisible by the smallest prime number 2.

2. Ans: A

$$\frac{2005^2+2\times 2005+1995+1995^2}{800}=\frac{(2005+1995)^2}{800}=\frac{400^2}{800}=20000.$$

3. Ans: D

   Since the equation $2y^2-8y-p=0$ has only one solution, its discriminant

$$(-8)^2-4\times 2\times(-p)=0.$$

   Thus $p=-8$.

4. Ans: B

   One easily checks that for $n\geq 10$, $n!$ will end with two zeros. Also,

$$\begin{aligned}&1!+2!+3!+4!+5!+6!+7!+8!+9!\\&=1+2+6+24+120+720+5040+40320+362880\\&=409113.\end{aligned}$$

   Thus the sum of the last two digits is 1+3=4.

5. Ans: B

$$5002^{2005\log_{5002}2005}=5002^{\log_{5002}2005^{2005}}=2005^{2005}.$$

6. Ans: A

   Since $RS=OP$, we have $RS=OR$. Hence, $\angle ROS=\angle RSO=12^\circ$. Then

$$\angle OQR=\angle ORQ=2\times 12^\circ=24^\circ.$$

   Thus,

$$\begin{aligned}\angle POQ&=180^\circ-\angle QOR-\angle ROS\\&=180^\circ-(180^\circ-24^\circ-24^\circ)-12^\circ\\&=36^\circ.\end{aligned}$$

7. Ans: E
Using Arithmetic Mean $\geq$ Geometric Mean, we have

$$\begin{aligned}\frac{1}{3}\left(\frac{16}{x}+\frac{108}{y}+xy\right) &\geq \sqrt[3]{\frac{16}{x}\times\frac{108}{y}\times xy}=12\\ \Longrightarrow\quad \frac{16}{x}+\frac{108}{y}+xy &\geq 36.\end{aligned}$$

Equality is achieved when $\frac{16}{x}=\frac{108}{y}=xy=\frac{36}{3}=12$, which is satisfied when $x=\frac{4}{3}$ and $y=9$. Thus the smallest possible value is 36.

8. Ans: E
We simply replace x by $-y$ and y by $-x$ to get

$$\begin{aligned}-x=\frac{2(-y)-1}{-y-1} &\Longrightarrow xy+x=-2y-1\\ &\Longrightarrow y=\frac{-1-x}{2+x}.\end{aligned}$$

9. Ans: B

$$\begin{aligned}\sqrt{2\left(1+\sqrt{1+\left(\frac{x^4-1}{2x^2}\right)^2}\right)} &= \sqrt{2\left(1+\sqrt{\frac{4x^4+x^8-2x^4+1}{4x^4}}\right)}\\ &= \sqrt{2\left(1+\sqrt{\left(\frac{4x^4+1}{2x^2}\right)^2}\right)}\\ &= \sqrt{2\left(\frac{2x^2+x^4+1}{2x^2}\right)}\\ &= \sqrt{\frac{(x^2+1)^2}{x^2}}\\ &= \frac{x^2+1}{x},\quad \text{since } x>0.\end{aligned}$$

10. Ans: C

$$\begin{aligned}x^2+y^2-2x+2y=2 &\Longleftrightarrow (x^2-2x+1)+(y^2+2y+1)=4\\ &\Longleftrightarrow (x-1)^2+(y+1)^2=2^2.\end{aligned}$$

Therefore, the points (x,y) in the Cartesian plane satisfying the equation consist of points on the circle centred at $(1,-1)$ and of radius 2. For a point (x,y) on the circle, it is easy to see that its distance $\sqrt{x^2+y^2}$ from the origin (0, 0) is greatest when it is the point in the fourth quadrant lying on the line passing through (0,0) and $(1, -1)$. In other words,

$$(x, y) = (1+\sqrt{2}, -1-\sqrt{2}), \text{ and } x^2+y^2 = (1+\sqrt{2})^2 + (-1-\sqrt{2})^2 = 6+4\sqrt{2}.$$

11. Ans: 193

$$\begin{aligned}(2+\sqrt{3})^4 &= [(2+\sqrt{3})^2]^2 = (4+4\sqrt{3}+3)^2 \\ &= (7+4\sqrt{3})^2 = 49+56\sqrt{3}+48 \approx 193.9.\end{aligned}$$

Thus, the greatest integer less than $(2+\sqrt{3})^4$ is 193.

12. Ans: 60

By Pythagoras' Theorem, $BC = \sqrt{169^2 - 156^2} = 65$. Let the perpendicular from C to AB meet AB at D. Then $\triangle ABC \sim \triangle ACD$. Thus,

$$\frac{CD}{AC} = \frac{BC}{AB} \Longrightarrow \frac{x}{156} = \frac{65}{169} \Longrightarrow x = 60.$$

13. Ans: 7

Let $a = 3^x - 27$ and $b = 5^x - 625$, so that the equation becomes

$$a^2 + b^2 = (a+b)^2 \Longrightarrow 2ab = 0 \Longrightarrow a = 0 \text{ or } b = 0.$$

When $a = 0$, we have $3^x - 27 = 0 \Longrightarrow x = 3$.
When $b = 0$, we have $5^x - 625 = 0 \Longrightarrow x = 4$.
Hence, the sum is $3 + 4 = 7$.

14. Ans: 12

Let x be the smallest integer. Then we have

$$x^3 \leq 2808 \leq x(x+6).$$

The first inequality implies that $x \leq 14$, while the second inequality implies that $x \geq 11$. If $x = 13$, then the product of the other integers is $2^3 \times 3^3 = 216$. This is impossible, since the smallest factor of 216 greater than 13 is 18 but $18^2 = 324 > 216$. Therefore, we must have $x = 12$. The other two integers are easily seen to be 13 and 18.

15. Ans: 1

Let $t = 2004$. Then the expression is equal to

$$\begin{aligned}&\frac{(t+1)^2(t^2-t+1)}{(t^2-1)(t^3+1)} \times \frac{(t-1)^2(t^2+t+1)}{t^3-1} \\ =& \frac{(t+1)^2(t^2-t+1)}{(t+1)(t-1)(t+1)(t^2-t+1)} \times \frac{(t-1)^2(t^2+t+1)}{(t-1)(t^2+t+1)} = 1.\end{aligned}$$

16. Ans: 6

$$f(x)+f(1-x) = \frac{1}{3^x+\sqrt{3}} + \frac{1}{3^{1-x}+\sqrt{3}} = \frac{\sqrt{3}}{3^x\sqrt{3}+3} + \frac{3^x}{3+3^x\sqrt{3}} = \frac{3^x+\sqrt{3}}{3+3^x\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Therefore,

$$\begin{aligned} &\sqrt{3}[f(-5)+f(6)] + \sqrt{3}[f(-4)+f(5)] + \sqrt{3}[f(-3)+f(4)] \\ &+ \sqrt{3}[f(-2)+f(3)] + \sqrt{3}[f(-1)+f(2)] + \sqrt{3}[f(0)+f(1)] \\ = &\sqrt{3}\Big(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\Big) = 6. \end{aligned}$$

17. Ans: 2049

$$16^5 + 2^{10} = (2^4)^5 + 2^{10} = (2^{10}+1)2^{10} = 1025 \times 1024 = 41 \times 5^2 \times 2^{10}.$$

Without loss of generality, we may assume that $A \leq B$. It follows that $1000 \leq A \leq 1024$, and a direct check shows that 1024 is the only number between 1000 and 1024 that divides $41 \times 5^2 \times 2^{10}$. Thus, $A = 1024$ and $B = 1025$. Therefore,

$$A + B = 1024 + 1025 = 2049.$$

18. Ans: 192
Let $AB = c$ cm, $BC = a$ cm and $CA = b$ cm. By the cosine rule,

$$a^2 = b^2 + c^2 - 2bc\cos 60^\circ = b^2 + c^2 - bc = (b-c)^2 + bc = 4^2 + bc = 16 + bc.$$

Let O be the centre of the circle. Then $\angle BOC = 2\angle A = 120^\circ$. Moreover, $OB = OC = 4$ cm. Therefore, by the cosine rule,

$$a^2 = 4^2 + 4^2 - 2 \times 4 \times 4\cos 120^\circ = 48 \Longrightarrow a = 4\sqrt{3}.$$

Then $bc = a^2 - 4 = 48 - 16 = 32$. Thus,

$$\text{area } (\triangle ABC) = \frac{1}{2}bc\sin 60^\circ = \frac{1}{2} \times 32 \times \frac{\sqrt{3}}{2} = 8\sqrt{3}.$$

Therefore, $x = 8\sqrt{3}$ and $x^2 = (8\sqrt{3})^2 = 192$.

19. Ans: 4

$$\begin{aligned} &c^2 = ab - 16 = (8-b)b - 16 = -b^2 + 8b - 16 \\ \Longrightarrow\ &c^2 + b^2 - 8b + 16 = 0 \Longrightarrow c^2 + (b-4)^2 = 0 \Longrightarrow c = 0 \ \text{ and } \ b = 4. \end{aligned}$$

Since $b = 4$, it follows that $a = 8 - 4 = 4$. Thus, $a + c = 4 + 0 = 4$. In this solution, we have used the fact that the sum of two non-negative numbers is zero only when both numbers are zero.

20. Ans: 8
Observe that

$$a_1^2 + (2a_2)^2 + \cdots + (8a_8)^2 \geq 1^2 + 2^2 + \cdots + 8^2 = 204.$$

So the only possiblility is that $a_i = 1$ for each $i = 1, 2, \ldots, 8$. Hence $a_1 + \cdots + a_8 = 8$.

21. Ans: 143
Observe that for any positive number x,

$$\frac{1}{\sqrt{x} + \sqrt{x+1}} = \frac{\sqrt{x+1} - \sqrt{x}}{(\sqrt{x+1})^2 - (\sqrt{x})^2} = \sqrt{x+1} - \sqrt{x}.$$

Thus we have

$$\begin{aligned} &\frac{1}{\sqrt{4} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{6}} + \cdots + \frac{1}{\sqrt{n} + \sqrt{n+1}} \\ &= (\sqrt{5} - \sqrt{4}) + (\sqrt{6} - \sqrt{5}) + \cdots + (\sqrt{n+1} - \sqrt{n}) \\ &= \sqrt{n+1} - \sqrt{4}. \end{aligned}$$

Hence, $\sqrt{n+1} - \sqrt{4} = 10 \Longrightarrow \sqrt{n+1} = 12 \Longrightarrow n + 1 = 144 \Longrightarrow n = 143$.

22. Ans: 211

$$\frac{1}{A} - \frac{1}{B} = \frac{192}{2005^2 - 2004^2} \Longleftrightarrow \frac{B - A}{A \times B} = \frac{192}{4009} \Longleftrightarrow 4009 \times (B - A) = 192 \times A \times B.$$

The prime factorization of 4009 is given by $4009 = 19 \times 211$. Since $192 = 2^6 \times 3$ and $B > A$, it follows that $A = 19$ and $B = 211$.

23. Ans: 108
Let the area of $\triangle AGF$ be S_1 and the area of $\triangle AGE$ be S_2. Let the area of $\triangle AMC$ be S, so that the area of $\triangle ABM$ is also S, since M is the midpoint of BC. Now,

$$\frac{S_1}{S} = \frac{AG \cdot AF}{AM \cdot AC} \quad \text{and} \quad \frac{S_2}{S} = \frac{AG \cdot AE}{AM \cdot AB}.$$

Therefore,

$$\frac{S_2}{S_1} = \frac{AG \cdot AE \cdot AM \cdot AC}{AM \cdot AB \cdot AG \cdot AF} = \frac{3}{4} \times 2 = \frac{3}{2}.$$

Note that $\frac{S_2}{S_1} = \frac{GE}{GF} = \frac{x}{72}$. Therefore, $\frac{3}{2} = \frac{x}{72} \Longrightarrow x = 108$.

24. Ans: 2

$$x = \frac{1}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}.$$

Then $x - 2 = \sqrt{3} \Longrightarrow x - \sqrt{3} = 2 \Longrightarrow x^2 - 4x + 1 = 0$ and $x^2 - 2\sqrt{3}x - 1 = 0$. Thus,

$$\begin{aligned} & x^6 - 2\sqrt{3}x^5 - x^4 + x^3 - 4x^2 + 2x - \sqrt{3} \\ &= x^4(x^2 - 2\sqrt{3}x - 1) + x(x^2 - 4x + 1) + x - \sqrt{3} \\ &= 0 + 0 + 2 + \sqrt{3} - \sqrt{3} = 2. \end{aligned}$$

25. Ans: 90

Since a, b are the roots of the equation $x^2 - (c+4)x + 4(c+2) = 0$, it follows that

$$a + b = c + 4 \quad \text{and} \quad \mathrm{ab} = 4(\mathrm{c}+2).$$

Then $a^2 + b^2 = (a+b)^2 - 2ab = (c+4)^2 - 8(c+2) \implies a^2 + b^2 = c^2$. Hence the triangle is right-angled, and $x = 90$.

26. Ans: 25

Observe that $\frac{1}{2}(a + b + |a - b|) = \max(a, b)$. Also, $x^2 > x - 1$ since $x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4} > 0$. Thus, we have

$$\frac{1}{2}[x^2 + x - 1 + |x^2 - (x-1)|] = \max(x^2, x-1) = x^2.$$

Thus the equation becomes $x^2 = 35x - 250 \Longrightarrow (x-25)(x-10) = 0 \Longrightarrow x = 25,\ 10$. Therefore, the largest value of x is 25.

27. Ans: 36

To determine the number of partitions, we just have to decide the number of ways of inserting a partition between two consecutive vowels if we insert a partition at all. Thus the total number of partitions is $4 \times 3 \times 3 = 36$.

28. Ans: 9011

For $n \geq 1$, $a_{n+1} = \dfrac{a_n}{1+na_n} \Longrightarrow \dfrac{1}{a_{n+1}} = \dfrac{1+na_n}{a_n} = \dfrac{1}{a_n} + n$. Therefore,

$$\begin{aligned} \frac{1}{a_{2005}} = \frac{1}{a_{2004}} + 2004 &= \frac{1}{a_{2003}} + 2003 + 2004 \\ &= \cdots \\ &= \frac{1}{a_1} + (1 + 2 + 3 + \cdots + 2004) = 1 + \frac{2004 \times 2005}{2} = 2009011. \end{aligned}$$

Hence, $\frac{1}{a_{2005}} - 2000000 = 9011$.

29. Ans: 401
By comparing the coefficients of x^k, it is easy to see that $a_k = k!$. Letting $x = 1$, we get

$$a_0 + a_1 + \cdots + a_k = (1+1)(1+2)(1+3)\cdots(1+k) = (k+1)!$$

Therefore, $a_0 + a_1 + \cdots + a_{k-1} = (k+1)! - k! = k \cdot k!$. Note that $2005 = 5 \times 401$, where 401 is a prime number. It is easy to see that a prime number p divides $k!$ only if $k \geq p$. Therefore, the smallest possible value of k is 401.

30. Ans: 1
Since the exponent on the left side of the given equation is non-zero if $x > 0$, it follows that the equation is possible only when

$$2x^3 - x^2 - x + 1 = 1 \Longrightarrow 2x^3 - x^2 - x = 0 \Longrightarrow x(x-1)(2x+1) = 0 \Longrightarrow x = 0, 1, -\frac{1}{2}.$$

Since $x > 0$, it follows that $x = 1$.

31. Ans: 6
We can rearrange the equation to get

$$x^2 - x(2+y) + y^2 - 2y = 0.$$

As a quadratic equation in the variable x, the discriminant of the above equation is given by

$$\text{Discriminant} = (2+y)^2 - 4(1)(y^2 - 2y) = 4 + 12y - 3y^2 = 16 - 3(y-2)^2.$$

Since x is an integer, the discriminant must be a perfect square and thus it is non-negative. Hence,

$$16 - 3(y-2)^2 \geq 0 \Longrightarrow (y-2)^2 \leq \frac{16}{3} \Longrightarrow |y-2| < \frac{4}{\sqrt{3}} < 3.$$

Since y is an integer, it follows that its only possible values are given by $y = 0, 1, 2, 3, 4$. When $y = 1, 3$, the discrimant is 12, which is not a perfect square. Thus, $y = 0, 2, 4$.
When $y = 0$, we have $x^2 - 2x = 0 \Longrightarrow x = 0, 2$.
When $y = 2$, we have $x^2 - 4x = 0 \Longrightarrow x = 0, 4$.
When $y = 4$, we have $x^2 - 6x + 8 = 0 \Longrightarrow x = 2, 4$.
Thus, there are 6 solutions, namely $(0,0), (2,0), (0,2), (4,2), (2,4), (4,4)$.

32. Ans: 133
From (i), it is easy to see that the whole ordered 7-tuple is determined by a_1 and a_2. It is easy to check that $a_6 = 3a_1 + 5a_2$. Thus by (ii), we have $3a_1 + 5a_2 = 2005$. It follows that a_1 is divisible by 5. Write $a_1 = 5k$. Then we have $15k + 5a_2 = 2005 \Longrightarrow 3k + a_2 = 401$. The possible values of a_2 are $\{2, 5, 8, \cdots, 398\}$, if both k

and a_2 are positive. Note that each of the above values of a_2 determines a unique positive integral value of a_1 satisfying the equation $3a_1 + 5a_2 = 2005$. Therefore, there are 133 such sequences.

33. Ans: 12
Let α be an integer satisfying the given equation, so that

$$
\begin{aligned}
&4\alpha^2 - (4\sqrt{3}+4)\alpha + \sqrt{3}n - 24 = 0 \\
\implies &4\alpha^2 - 4\alpha - 24 = \sqrt{3}(4\alpha - n).
\end{aligned}
$$

Since $\sqrt{3}$ is irrational, it follows that

$$4\alpha^2 - 4\alpha - 24 = 0 \text{ and } 4\alpha - n = 0.$$

Substituting $\alpha = \frac{n}{4}$ into the first equation, we have

$$
\begin{aligned}
&4(\frac{n}{4})^2 - 4(\frac{n}{4}) - 24 = 0 \\
\implies &n^2 - 4n - 96 = 0 \\
\implies &(n-12)(n+8) = 0 \\
\implies &n = 12, -8.
\end{aligned}
$$

Since n is a positive integer, we have $n = 12$.

34. Ans: 2
Subtracting the second equation from the first one, we have

$$
\begin{aligned}
&xy + yz = 255 - 224 = 31 \\
\implies &y(x+z) = 31.
\end{aligned}
$$

Since 31 is a prime number, we have $y = 1$ and $x + z = 31$. Together with the first given equation, we have

$$
\begin{aligned}
&x \cdot 1 + x(31 - x) = 255 \\
\implies &(x-15)(x-17) = 0 \\
\implies &x = 15 \text{ or } 17.
\end{aligned}
$$

When $x = 15$, $y = 1$, we have $z = 16$.
When $x = 17$, $y = 1$, we have $z = 14$.
Therefore, there are two such solutions, namely $(15, 1, 16)$ and $(17, 1, 14)$.

35. Ans: 210

Let $abcd = 1000a+100b+10c+d$ be a four-digit number divisible by 7. In particular, $1 \leq a \leq 9$. We are given that $dbca = 1000d + 100b + 10c + a$ is also divisible by 7. Observe that

$$abcd - dbca = 999a - 999d = 999(a - d).$$

Since 7 does not divide 999, it follows that 7 must divide $a - d$. Thus $a \equiv d$ mod 7. Since 7 divides 1001, we have

$$\begin{aligned} & 1000a + 100b + 10c + d \equiv 0 \mod 7 \\ \Longrightarrow & -a + 100b + 10c + d \equiv 0 \mod 7 \\ \Longrightarrow & 100b + 10c \equiv 0 \mod 7 \\ \Longrightarrow & 10(10b + c) \equiv 0 \mod 7 \\ \Longrightarrow & 10b + c \equiv 0 \mod 7. \end{aligned}$$

Therefore we must have $a \equiv d$ mod 7 and $10b + c \equiv 0$ mod 7.

Conversely, by reversing the above arguments, one easily sees that if $a \equiv d$ mod 7 and $10b + c \equiv 0$ mod 7, then both $abcd$ and $dbca$ are divisible by 7.

Now there are 14 pairs of (a, b) satisfying $a \equiv d$ mod 7, which are

$$(1, 1), (2, 2), (3, 3), \ldots, (9, 9), (1, 8), (2, 9), (7, 0), (8, 1), (9, 2).$$

Since $0 \leq 10b + c \leq 99$, there are 15 pairs of (b, c) satisfying $10b + c \equiv 0$ mod 7 (including $(0, 0)$). In fact, this is equal to the number of non-negative integers less that 100 that are divisible by 7. Therefore, there are $14 \times 15 = 210$ numbers with the required properties.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2005

(Senior Section, Special Round Solutions)

1. Since the sums of the digits of the two numbers are the same, they leave the same remainder when they are divided by 9. Thus their difference is divisible by 9. So the possible answers are 36 or 81. It is easy to see that both cases are possible: For example: $(645, 564)$ and $(218, 182)$.

2. Let M and N be points on AB and AD so that $MI \parallel BC$ and $NJ \parallel DC$. Let $a = DI = CF$, and $b = BJ = CE$. Then

$$\triangle KJB \simeq \triangle KIM, \qquad \triangle LJN \simeq \triangle LID$$
$$\triangle NJE \simeq \triangle DCE, \qquad \triangle BCF \simeq \triangle MIF$$

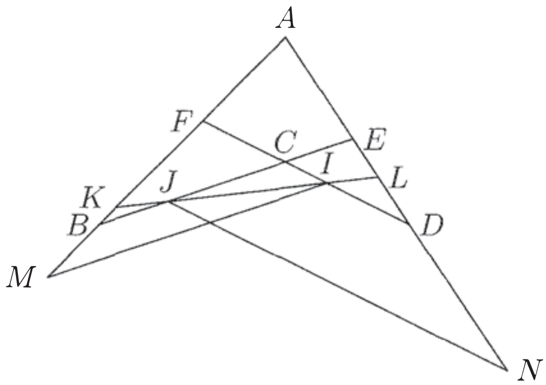
Thus

$$\frac{KJ}{b} = \frac{KJ + JI}{MI}, \qquad \frac{JI + IL}{NJ} = \frac{IL}{a}$$
$$\frac{NJ}{b + CJ} = \frac{a + IC}{b}, \qquad \frac{b + CJ}{a} = \frac{MI}{a + IC}$$

Multiplying the four equations we get

$$\frac{(KJ)(JI) + (KJ)(IL)}{ab} = \frac{(IL)(JI) + (KJ)(IL)}{ab}$$

which yields $KJ = IL$.



3. Suppose that this result is not true. Observe that S has total of $\binom{10}{2} = 45$ 2-element subsets. Let $S_i = \{x_{i1}, x_{i2}\}$, $i = 1, 2, \ldots, 45$, be the 45 2-element subsets of S and $s_i = x_{i1} + x_{i2}$. Then by the assumption, the 45 values s_i are mutually distinct. Since $3 \leq s_i \leq 47$, and there are exactly 45 numbers from 3 through 47, we have

$\{s_i : i = 1, \ldots, 45\} = \{3, 4, \ldots, 47\}$. Since there is pair summing to 3 and a pair summing to 47, the numbers $1, 2, 23, 24 \in S$. But then $1 + 24 = 2 + 23$ gives rise to a contradiction.

4. Let $m = 1000100000000$. Let $M_n = (m + n)!$, where n is an integer such that $0 \leq n \leq 10000$. Observe that if $M_n = \overline{abcde\ldots}$, where $a, b, c, d, \ldots$ are the digits of M_n, then the first 4 digits of M_{n+1} are (i) $\overline{abcd}$, in which case the fifth digit is $a + e > e$, (ii) the first four digits of $\overline{abcd} + 1$ or (iii) the first four digits of $\overline{abcd} + 2$. The last case can happen only when $a = 9$. Therefore we see that if $a < 9$, among the numbers M_0, M_1, …, M_{10}, there is at least one for which the first four digits are the first four digits of $\overline{xyzw} + 1$ where $\overline{xyzw}$ are the first four digits of M_0. It follows that the first four digits of the numbers numbers M_0, M_1, …, M_{10000} includes all of $1000, 1001, \ldots, 8999$. Hence the answer is yes.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2005

(Open Section, Round 1)

Wednesday, 1 June 2005 0930-1200

Answer ALL 25 questions.
Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.
No steps are needed to justify your answers.
Each question carries 1 mark.
No calculators are allowed.

1. Find the last three digits of $9^{100} - 1$.

2. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have radii 3 and 7 respectively. The circles intersect at distinct points A and B. A point P outside $\mathcal{C}_2$ lies on the line determined by A and B at a distance of 5 from the center of $\mathcal{C}_1$. Point Q is chosen on $\mathcal{C}_2$ so that PQ is tangent to $\mathcal{C}_2$ at Q. Find the length of the segment PQ.

3. Find the largest positive integer n such that $n!$ ends with exactly 100 zeros.

4. Find the largest value of $3k$ for which the following equation has a real root:
$$\sqrt{x^2 - k} + 2\sqrt{x^2 - 1} = x.$$

5. Find the last two digits (in order) of $7^{3^{2005}}$.

6. From the first 2005 natural numbers, k of them are arbitrarily chosen. What is the least value of k to ensure that there is at least one pair of numbers such that one of them is divisible by the other?

7. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where a_i are nonnegative integers for $i = 0, 1, 2, \ldots, n$. If $f(1) = 21$ and $f(25) = 78357$, find the value of $f(10)$.

8. If x is a real number that satisfies
$$\left\lfloor x + \frac{11}{100} \right\rfloor + \left\lfloor x + \frac{12}{100} \right\rfloor + \cdots + \left\lfloor x + \frac{99}{100} \right\rfloor = 765,$$

find the value of $\lfloor 100x \rfloor$. Here $\lfloor a \rfloor$ denotes the largest integer $\leq a$.

9. The function $f(n)$ is defined for all positive integer n and take on non-negative integer values such that $f(2) = 0$, $f(3) > 0$ and $f(9999) = 3333$. Also, for all m, n,

$$f(m+n) - f(m) - f(n) = 0 \quad \text{or} \quad 1.$$

Determine $f(2005)$.

10. It is known that the 3 sides of a triangle are consecutive positive integers and the largest angle is twice the smallest angle. Find the perimeter of this triangle.

11. A triangle $\triangle ABC$ is inscribed in a circle of radius 1, with $\angle BAC = 60°$. Altitudes AD and BE of $\triangle ABC$ intersect at H. Find the smallest possible value of the length of the segment AH.

12. Let $\mathbb{N}$ be the set of all positive integers. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(m + n) = f(f(m) + n)$ for all $m, n \in \mathbb{N}$, and $f(6) = 2$. Also, no two of the values $f(6), f(9), f(12)$ and $f(15)$ coincide. How many three-digit positive integers n satisfy $f(n) = f(2005)$?

13. Let f be a real-valued function so that

$$f(x, y) = f(x, z) - 2f(y, z) - 2z$$

for all real numbers x, y and z. Find $f(2005, 1000)$.

14. Let $a_1 = 2006$, and for $n \geq 2$,

$$a_1 + a_2 + \cdots + a_n = n^2 a_n.$$

What is the value of $2005\, a_{2005}$?

15. Find the smallest three-digit number n such that if the three digits are a, b and c, then

$$n = a + b + c + ab + bc + ac + abc.$$

16. Let $a_1 = 21$ and $a_2 = 90$, and for $n \geq 3$, let a_n be the last two digits of $a_{n-1} + a_{n-2}$. What is the remainder of $a_1^2 + a_2^2 + \cdots + a_{2005}^2$ when it is divided by 8.

17. Find the smallest two-digit number N such that the sum of digits of $10^N - N$ is divisible by 170.

18. Find the least n such that whenever the elements of the set $\{1, 2, \ldots, n\}$ are coloured red or blue, there always exist x, y, z, w (not necessarily distinct) of the same colour such that $x + y + z = w$.

19. Let x and y be positive integers such that $\frac{100}{151} < \frac{y}{x} < \frac{200}{251}$. What is the minimum value of x?

20. Find the maximum positive integer n such that

$$n^2 \leq 160 \times 170 \times 180 \times 190.$$

21. Find the number of positive integers n such that

$$n + 2n^2 + 3n^3 + \cdots + 2005n^{2005}$$

is divisible by $n - 1$.

22. Given ten 0's and ten 1's, how many 0-1 binary sequences can be formed such that no three or more than three 0's are together? For example, 01001001010011101011 is such a sequence, but the sequence 01001000101001110111 does not satisfy this condition.

23. In triangle ABC, $AB = 28$, $BC = 21$ and $CA = 14$. Points D and E are on AB with $AD = 7$ and $\angle ACD = \angle BCE$. Find the length of BE.

24. Four points in the order A, B, C, D lie on a circle with the extension of AB meeting the extension of DC at E and the extension of AD meeting the extension of BC at F. Let EP and FQ be tangents to this circle with points of tangency P and Q respectively. Suppose $EP = 60$ and $FQ = 63$. Determine the length of EF.

25. A pentagon $ABCDE$ is inscribed in a circle of radius 10 such that BC is parallel to AD and AD intersects CE at M. The tangents to this circle at B and E meet the extension of DA at a common point P. Suppose $PB = PE = 24$ and $\angle BPD = 30°$. Find BM.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2005

(Open Section, Special Round)

Saturday, 2 July 2005 0900– 1330

Attempt as many questions as you can.
No calculators are allowed.
Show the steps in your calculations.
Each question carries 10 marks.

1. An integer is square-free if it is not divisible by a^2 for any integer $a > 1$. Let S be the set of positive square-free integers. Determine, with justification, the value of

$$\sum_{k \in S} \left\lfloor \sqrt{\frac{10^{10}}{k}} \right\rfloor,$$

where $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example $\lfloor 2.5 \rfloor = 2$.

2. Let G be the centroid of $\triangle ABC$. Through G draw a line parallel to BC and intersecting the sides AB and AC at P and Q, respectively. Let BQ intersect GC at E and CP intersect GB at F. Prove that triangles ABC and DEF are similar.

3. Let a, b, c be real numbers satisfying $a < b < c$, $a + b + c = 6$ and $ab + bc + ac = 9$. Prove that $0 < a < 1 < b < 3 < c < 4$.

4. Place 2005 points on the circumference of a circle. Two points P, Q are said to form a pair of *neigbours* if the chord PQ subtends an angle of at most $10°$ at the centre. Find the smallest number of pairs of neighbours.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2005

(Open Section, Round 1 Solutions)

1. Ans: 000

$$9^{100}-1=(1-10)^{100}-1=1-\binom{100}{1}10^1+\binom{100}{2}10^2-\cdots+\binom{100}{100}10^{100}-1=1000k$$

for some integer k. Thus, the last three digits are 000.

2. Ans: 4

P lies on the radical axis of $\mathcal{C}_1$ and $\mathcal{C}_2$ and hence has equal power with respect to both circles. Now

$$\begin{aligned} PQ^2 &= \text{ Power of } P \text{ with respect to } \mathcal{C}_2 \\ &= \text{ Power of } P \text{ with respect to } \mathcal{C}_1 \\ &= 5^2-3^2=16. \end{aligned}$$

3. Ans: 409

If $n!$ ends with exactly 100 zeros, then in the prime factorization of $n!$, the prime factor 5 occurs exactly 100 times (we need not worry about the prime factor 2 since it will occur more than 100 times). The number of times that 5 occurs in $n!$ is given by

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots.$$

Also the factor 5 occurs 24 times in 100!. Thus the answer is about 400. Now

$$\left\lfloor \frac{400}{5} \right\rfloor + \left\lfloor \frac{400}{5^2} \right\rfloor + \left\lfloor \frac{400}{5^3} \right\rfloor + \cdots = 80+16+3=99.$$

It follows that 400! ends with exactly 99 zeros. Thus the answer is 409.

4. Ans: 4

For the equation to have a solution, k must be nonnegative, for if $k<0$, then

$$\sqrt{x^2-k}+2\sqrt{x^2-1} \geq \sqrt{x^2-k} > x.$$

Rewrite the given equation as $2\sqrt{x^2-1}=x-\sqrt{x^2-k}$. By squaring we obtain $2x^2+k-4=-2x\sqrt{x^2-k}$. Squaring again and solving for x, we obtain

$$x^2=\frac{(k-4)^2}{8(2-k)}.$$

Thus $0 \leq k < 2$ and the only possible solution is $x = \frac{4-k}{\sqrt{8(2-k)}}$. Substitute this value of x into the given equation and multiplying throughout by $\sqrt{8(2-k)}$, we obtain

$$\sqrt{(k-4)^2 - 8k(2-k)} + 2\sqrt{(k-4)^2 - 8(2-k)} = 4 - k.$$

Hence, $|3k-4| + 2k = 4 - k$ and so $|3k-4| = -(3k-4)$ which holds if and only if $3k - 4 \leq 0$; i.e. $3k \leq 4$. Therefore, the maximum value of $3k$ is 4.

5. Ans: 43

Note that $7^4 = 1 \pmod{100}$. Now

$$3^{2005} = (-1)^{2005} = -1 = 3 \pmod 4.$$

Hence $3^{2005} = 4k + 3$ for some positive integer k. Thus

$$7^{3^{2005}} = 7^{4k+3} = 7^3 = 43 \pmod{100}.$$

6. Ans: 1004

Take any set 1004 numbers. Write each number in the form $2^{a_i}b_i$, where $a_i \geq 0$ and b_i is odd. Thus obtaining 1004 odd numbers $b_1, \ldots, b_{1004}$. Since there are 1003 odd numbers in the first 2005 natural numbers, at least two of the odd numbers are the same, say $b_i = b_j$. Then $2^{a_i}b_i \mid 2^{a_j}b_j$ if $2^{a_i}b_i < 2^{a_i}b_i$. The numbers $1003, 1004, \ldots, 2005$ are 1003 numbers where there is no pair such that one of them is divisible by the other. So the answer is 1004.

7. Ans: 5097

We use the fact that every positive integer can be expressed uniquely in any base ≥ 2. As $f(1) = 21$, we see that the coefficients of $f(x)$ are nonnegative integers less than 25. Therefore $f(25) = 78357$, when written in base 25, allows us to determine the coefficients. Since

$$78357 = 7 + 9 \times 25 + 5 \times 25^3,$$

we obtain $f(x) = 7 + 9x + 5x^3$. Hence $f(10) = 5097$.

8. Ans: 853

First observe that $\lfloor x + \frac{k}{100} \rfloor = \lfloor x \rfloor$ or $\lfloor x \rfloor + 1$ for $11 \leq k \leq 99$. Since there are 89 terms on the left-hand side of the equation and $89 \times 8 < 765 < 89 \times 9$, we deduce that $\lfloor x \rfloor = 8$. Now suppose $\lfloor x + \frac{k}{100} \rfloor = 8$ for $11 \leq k \leq m$ and $\lfloor x + \frac{k}{100} \rfloor = 9$ for $m + 1 \leq k \leq 99$. Then

$$8(m - 10) + 9(99 - m) = 765,$$

which gives $m = 46$. Therefore $\lfloor x + \frac{46}{100} \rfloor = 8$ and $\lfloor x + \frac{47}{100} \rfloor = 9$, which imply $8 \leq x + \frac{46}{100} < 9$ and $9 \leq x + \frac{47}{100} < 10$ respectively. The inequalities lead to

$7.54 \le x < 8.54$ and $8.53 \le x < 9.53$. Consequently, we see that $8.53 \le x < 8.54$. Hence we conclude that $\lfloor 100x \rfloor = 853$.

9. Ans: 668

The given relation implies that

$$f(m+n) \ge f(m) + f(n).$$

Putting $m = n = 1$ we obtain $f(2) \ge 2f(1)$. Since $f(2) = 0$ and $f(1)$ is non-negative, $f(1) = 0$. Next, since

$$f(3) = f(2+1) = f(2) + f(1) + \{0 \quad \text{or} \quad 1\} = \{0 \quad \text{or} \quad 1\},$$

where $f(3) > 0$, it follows that $f(3) = 1$. Now, it can be proved inductively that $f(3n) \ge n$ for all n. Also, if $f(3n) > n$ for some n, then $f(3m) > m$ for all $m \ge n$. So since $f(3 \times 3333) = f(9999) = 3333$, it follows that $f(3n) = n$ for $n \le 3333$. In particular $f(3 \times 2005) = 2005$. Consequently,

$$\begin{aligned} 2005 &= f(3 \times 2005) \\ &\ge f(2 \times 2005) + f(2005) \\ &\ge 3f(2005) \end{aligned}$$

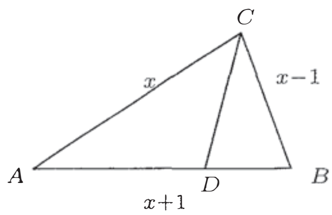
and so $f(2005) \le 2005/3 < 669$. On the other hand,

$$f(2005) \ge f(2004) + f(1) = f(5 \times 668) = 668.$$

Therefore $f(2005) = 668$.

10. Ans: 15

Let $\angle C = 2\angle A$ and CD the bisector of $\angle C$. Let $BC = x - 1, CA = x$ and $AB = x + 1$. Then $\triangle ABC$ is similar to $\triangle CBD$. Thus $BD/BC = BC/AB$ so that $BD = (x-1)^2/(x+1)$. Also $CD/AC = CB/AB$ so that $AD = CD = x(x-1)/(x+1)$.



As $AB = AD + BD$, we have $x(x-1)/(x+1) + (x-1)^2/(x+1) = x + 1$. Solving this, the only positive solution is $x = 5$. Thus the perimeter of the triangle is $4 + 5 + 6 = 15$.

11. Ans: 1

$AE = AB\cos 60°$. $\angle AHE = \angle ACB$. Hence

$$AH = AE/\sin\angle AHE = AB\cos 60°/\sin\angle ACB = 2R\cos 60°,$$

where $R = 1$ is the radius of the circle. Therefore, $AH = 1$.

12. Ans: 225

Since $f(6+n) = f(f(6)+n) = f(2+n)$ for all n, the function f is periodic with period 4 starting from 3 onwards. Now $f(6), f(5) = f(9), f(4) = f(12)$ and $f(3) = f(15)$ are four distinct values. Thus in every group of 4 consecutive positive integers ≥ 3, there is exactly one that is mapped by f to $f(2005)$. Since the collection of three-digit positive integers can be divided into exactly 225 groups of 4 consecutive integers each, there are 225 three-digit positive integers n that satisfies $f(n) = f(2005)$.

13. Ans: 5

Setting $x = y = z$, we see that

$$f(x,x) = f(x,x) - 2f(x,x) - 2x$$

for all x. Hence $f(x,x) = -x$ for all x. Setting $y = x$ gives

$$f(x,x) = f(x,z) - 2f(x,z) - 2z$$

for all x and z. Hence $f(x,z) = -f(x,x) - 2z = x - 2z$ for all x and z. Therefore, $f(2005, 1000) = 5$.

14. Ans: 2

$$a_n = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i = n^2 a_n - (n-1)^2 a_{n-1}.$$

This gives $a_n = \frac{n-1}{n+1} a_{n-1}$. Thus,

$$a_{2005} = \frac{2004}{2006} a_{2004} = \cdots = \frac{(2004)(2003)\cdots(1)}{(2006)(2005)\cdots(3)} a_1 = \frac{2}{2005}.$$

15. Ans: 199

From $100a + 10b + c = a + b + c + ab + bc + ac + abc$, we get $(99 - b - c - bc)a = b(c-9)$. Note that LHS ≥ 0 (since $b + c + bc = (1+b)(1+c) - 1 \leq 99$) and RHS ≤ 0. Thus we must have $b = c = 9$. Since $a \geq 1$, $n = 199$ is the answer.

16. Ans: 1

Note that the value of $x^2 \pmod 8$ is completely determined by the value of $x \pmod 4$. Furthermore, $a_n \equiv a_{n-1} + a_{n-2} \pmod{100} \equiv a_{n-1} + a_{n-2} \pmod 4$. Thus the sequence $(a_n \pmod 4)_{n\in\mathbb{Z}^+}$ equals $(1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, 1, 1, 2 \ldots)$, which has a cycle of length 6. Also, the sum of the squares in one cycle is $0 \pmod 8$. Now $2005 = 6 \times 334 + 1$. Thus the required answer is 1.

17. Ans: 20

Let $N = 10a + b$. When $b \neq 0$, $10^N - N$ is a N-digit number whose first $N - 2$ digits are 9 and the last two digits are $(9 - a)$ and $(10 - b)$. Thus the sum of digits of $10^N - N$ equals $9(N - 2) + 19 - (a + b) = 89a + 8b + 1$. Thus, we need

$$\begin{aligned} a &\equiv 1 \pmod 2; \\ -a + 3b &\equiv -1 \pmod 5; \\ 4a + 8b &\equiv -1 \pmod{17}. \end{aligned}$$

$a = 1$ has no solution, but $a = 3, b = 9$ is a solution.

When $b = 0$, the sum of digits is $9(N - 2) + (10 - a) = 89a - 8$. Thus a is even. $a = 2$ is a solution. Thus the answer is 20 since it is < 39.

18. Ans: 11

It can easily be checked that if $1, 2, 9, 10$ are red and $3, 4, 5, 6, 7, 8$ are blue, then there do not exist x, y, z, w of the same colour such that $x + y + z = w$. So $n \geq 11$.

Suppose there exists a colouring (for $n = 11$) such that there do not exist x, y, z, w of the same colour such that $x + y + z = w$. Then $1, 9$ must be of a different colour from 3. Thus say 1 and 9 are red, and 3 is blue. Since $7 + 1 + 1 = 9$, 7 must then be blue. Now, $2 + 3 + 3 = 7$, so that 2 is red, and hence 4 $(= 1 + 1 + 2)$ is blue. But $11 = 3 + 4 + 4 = 9 + 1 + 1$, a contradiction.

19. Ans: 3

The inequality can be transformed to

$$\frac{302y}{200} > x > \frac{251y}{200}.$$

We first need to the minimum positive integer y such that the interval $\left(\frac{302y}{200}, \frac{251y}{200}\right)$ contains an integer. This minimum value of y is 2. When $y = 2$, this interval contains only one integer, that is 3. Thus the answer is 3.

20. Ans: 30499

Let $f = 160 \times 170 \times 180 \times 190$. Observe that $f = (175^2 - 15^2)(175^2 - 5^2)$. Let $x = 175^2 - 125$. Then $f = (x - 100)(x + 100) = x^2 - 10000 < x^2$. Since $x = 175^2 - 125$

and $f-(x-1)^2 = x^2-1000-(x-1)^2 = 2x-10001$, we have $f > (x-1)^2$. Thus the minimum positive integer n such that $n^2 < f$ is $x-1 = 175^2-126 = 30499$.

21. Ans: 16

Let $n+\cdots+2005n^{2005} = f(n)(n-1)+a$ where $f(n)$ is a polynomial of n. If $n=1$, we have $a = 1+2+\cdots+2005 = 2005\times 1003$. Observe that $n+\cdots+2005n^{2005}$ is divisible by $n-1$ if and only if $a = 2005\times 1003$ is divisible by $n-1$. The prime factorization of a is $a = 5\times 401\times 17\times 59$. So a has only $2^4 = 16$ factors. Thus there are 16 possible positive integers n such that $n+\cdots+2005n^{2005}$ is divisible by $n-1$.

22. Ans: 24068

Consider the sequence of ten 1's. There are eleven spaces between two 1's, or before the leftmost 1 or after the rightmost 1. For each of these spaces, we can put either two 0's (double 0's) or only one 0 (single 0). If there are exactly k doubles 0's, then there are only $10-2k$ single 0's. The number of ways for this to happen is $\binom{11}{k}\binom{11-k}{10-2k}$. Thus the answer is

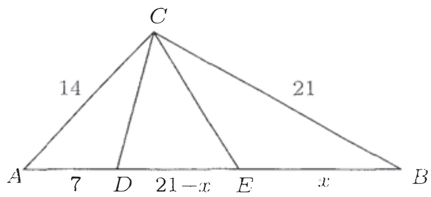
$$\sum_{k=0}^{5}\binom{11}{k}\binom{11-k}{10-2k} = 24068.$$

23. Ans: 12

Let $BE = x$. By cosine rule,

$$\cos B = \frac{21^2+28^2-14^2}{2(21)(28)} = \frac{7}{8} \quad\text{and}\quad \cos A = \frac{14^2+28^2-21^2}{2(14)(28)} = \frac{11}{16}.$$

Thus $CD^2 = 14^2+7^2-2(14)(7)(\frac{11}{16}) = \frac{441}{4}$ so that $CD = \frac{21}{2}$.



Also

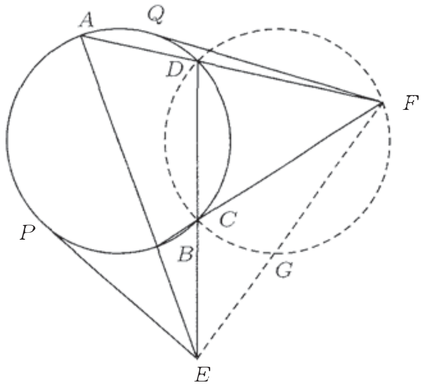
$$\cos\angle ACD = \frac{14^2+(\frac{21}{2})^2-7^2}{2(14)(\frac{21}{2})} = \frac{7}{8}.$$

Thus $\angle B = \angle ACD = \angle BCE$ so that triangle BEC is isosceles. Therefore,

$$x = \frac{BC}{2\cos B} = \frac{21}{2}\frac{8}{7} = 12.$$

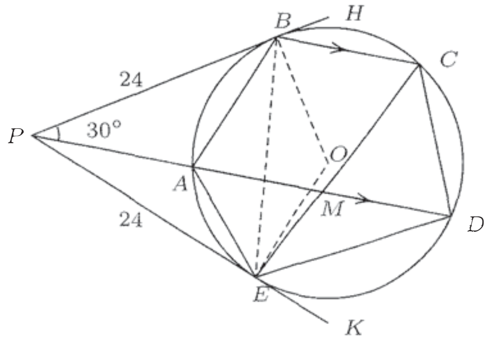
24. Ans: 87

Let the circumcircle of $\triangle CDF$ meet the line EF at G. Then G, C, D, F are concyclic. Now $\angle EBC = \angle ADC = \angle CGF$ so that E, B, C, G are also concyclic. Thus, $EP^2 = EB \cdot EA = EC \cdot ED = EG \cdot EF$ and $FQ^2 = FD \cdot FA = FC \cdot FB = FG \cdot FE$. Therefore, $EP^2 + FQ^2 = EG \cdot EF + FG \cdot FE = (EG + FG) \cdot EF = EF^2$. Consequently, $EF = \sqrt{60^2 + 63^2} = 87$.



25. Ans: 13

Let O be the centre of the circle. First we show that P, B, O, M, E lie on a common circle. Clearly P, B, O, E are concyclic. As $\angle BPM = \angle HBC = \angle BEM = 30°$, points P, B, M, E are also concyclic. Thus P, B, O, M, E all lie on the circumcircle of $\triangle PBE$.



Since $\angle PBO = 90^{\bullet}$, PO is a diameter of this circle and $PO = \sqrt{24^2 + 10^2} = 26$. This circle is also the circumcircle of $\triangle PBM$. Therefore, $BM = 26 \sin 30° = 13$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2005

(Open Section, Special Round Solutions)

1. We shall show that in general for any $n \in \mathbb{N}$,

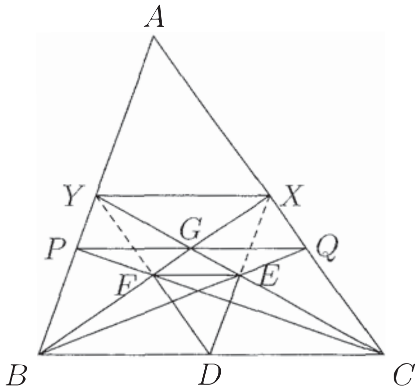
$$\sum_{k\in S}\left\lfloor\sqrt{\frac{n}{k}}\right\rfloor = n.$$

For any $k \in S$, note that $\lfloor\sqrt{n/k}\rfloor = j$ if and only if j is the largest integer such that $j^2k \leq n$. If $S_k = \{1^2k, 2^2k.\dots, j^2k\}$, where j is the largest integer such that $j^2k \leq n$, then $\lfloor\sqrt{n/k}\rfloor = |S_k|$. Also if k and m are distinct square free integers, then $S_k \cap S_m = \emptyset$. Hence

$$\sum_{k\in S}\left\lfloor\sqrt{\frac{n}{k}}\right\rfloor = \sum_{k\in S}|S_k| = \left|\bigcup_{k\in S} S_k\right|.$$

Since every integer can be uniquely written as j^2k where k is square free, $\cup_{k\in S} = \{1, 2, \dots, n\}$. This completes the proof.

2. First we note the centroid divides each median in the ratio 1:2. Let X be the midpoint of AC and Y be the midpoint of AB. Then $XQ/QC = XG/GB = 1/2$. So, by Ceva's theorem, X, E, D are collinear. Likewise, D, F, Y are collinear. Therefore $DE \parallel BA$ and E is the midpoint of CY. Similarly, $DF \parallel CA$ and F is the midpoint of DY. Thus $EF \parallel XY \parallel CB$. Since the corresponding sides are parallel, the two triangles are similar.



3. Firstly, $a, b, c \neq 0$ since if, for example, $c = 0$, then $a + b = 6$ and $ab = 9$ imply that $a = b = 3$, a contradiction. Also, a, b, c satisfies $x^3 - 6x^2 + 9x - abc = 0$, ie. $(x-3)^2 = abc/x$. Since $(x-3)^2 \geq 0$ for all x, we see that a, b, c are all of the same sign, and thus $a, b, c > 0$ as $a + b + c = 6$.

   Now the graph $y = x^3 - 6x^2 + 9x - abc$ has stationary points at $x = 1$ and $x = 3$, and cuts the x-axis at $x = a, b, c$, so that $a \in (-\infty, 1)$, $b \in (1, 3)$ and $c \in (3, \infty)$. Finally, if $c > 4$, then $ab = (c-3)^2 > 1$, so that $ab + bc + ac > 1 + 4(b + 1/b)$. But $x + 1/x$ is increasing for $x \in (1, \infty)$. Thus $1 + 4(b + 1/b) > 9$, a contradiction.

   Second solution: As in the first solution, a, b and c satisfy the equation $x^3 - 6x^2 + 9x - abc = 0$, or $x(x-3)^2 = abc$. Substitute $x = c$ to get $c(c-3)^2 = abc$ or $(c-3)^2 = ab$. Now, assume $c \geq 4$. This implies that $ab \geq 1$, which in turn implies $a + b > 2$ (via AM-GM). Thus $c < 4$, a contradiction. Hence $c < 4$. We get $abc = c(c-3)^2 < 4$. ($c > 2$, since $a + b + c = 6$)

   For the equation $f(x) = abc$, where $abc < 4$, $f(0) = 0$, $f(1) = 4$ implies one root between 0 and 1 $f(1) = 4$, $f(3) = 0$ implies one root between 1 and 3. $f(3) = 0$, $f(4) = 4$ implies one root between 3 and 4.

4. Let P, Q be a pair of neighbours Suppose P has k neighbours who are not neighbours of Q and Q has ℓ neighbours who are not neightbours of P, with $k \geq \ell$. Then by moving P to a point very close to Q the number of neighbouring pairs does not increase and the set of neighbours of P and Q coincide. Thus by repeating this procedure, we see that the minimum is attained when the points are divided into s clusters, $s \leq 35$, such that two points are neighbours iff they are in the same cluster. Thus, if n_i is the number of points in cluster i, then

$$\text{minimum} \geq \sum \binom{n_i}{2} \geq 25\binom{57}{2} + 10\binom{58}{2} = 56430.$$

   The last inequality follows from the following: If the largest of the n_i's, say a, is > 58, then the smallest, say b, is ≤ 57 and $\binom{a}{2} + \binom{b}{2} \geq \binom{a-1}{2} + \binom{b+1}{2}$ which can be proved by direct computation. This minimum is attained when there are 35 clusters, 25 with 57 points and 10 with 58 points.

2005 Senior round 2, Q3 solution: Observe that S has a total of $\binom{10}{2} = 45$ 2-element subsets, giving rise to a total of 45 sums. The values of these sums ranges from 3 to 47, both inclusive. If the values 3 and 47 are both present, then 1, 2, 23, 24 are in S. Then the sets $\{1, 24\}$ and $\{2, 23\}$ both have the sum 25. If not, then there are only 44 possible values for the sums. Thus, by the pigeonhole principle, two of sums are equal.

SINGAPORE MATHEMATICAL OLYMPIADS 2006

Roger Poh Kheng Siong
Tay Tiong Seng
To wing Keung
Yang Yue

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Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Junior Section)

Tuesday, 30 May 2006 0930-1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter only the letters (A, B, C, D, or E) corresponding to the correct answers in the answer sheet.

For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

1. What are the last two digits of $1 \times 2 \times 3 \times 4 \times \cdots \times 2004 \times 2005 \times 2006$?

   (A) 00; (B) 20; (C) 30; (D) 50; (E) 60.

2. Let x be a real number. What is the minimum value of $x^2 - 4x + 3$?

   (A) -3; (B) -1; (C) 0; (D) 1; (E) 3.

3. James calculates the sum of the first n positive integers and finds that the sum is 5053. If he has counted one integer twice, which one is it?

   (A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

4. Which of the following is a possible number of diagonals of a convex polygon?

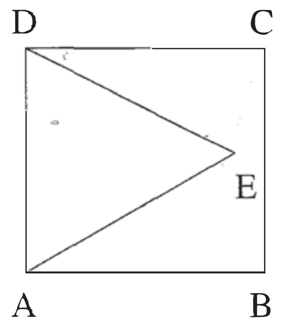
   (A) 21; (B) 32; (C) 45; (D) 54; (E) 63.

5. What is the largest positive integer n satisfying $n^{200} < 5^{300}$?

   (A) 9; (B) 10; (C) 11; (D) 12; (E) 13.

6. The diagram shows an equilateral triangle ADE inside a square $ABCD$. What is the value of

$$\frac{\text{area of } \Delta ADE}{\text{area of } \Delta DEC}?$$



(A) $\frac{\sqrt{3}}{4}$; (B) $\frac{1}{4}$; (C) $\frac{\sqrt{3}}{2}$; (D) $\sqrt{3}$; (E) 2.

7. What is the value of

$$(x+1)(x+2006)[\frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \ldots + \frac{1}{(x+2005)(x+2006)}]?$$

(A) $x + 2004$; (B) 2005; (C) $x + 2006$; (D) 2006; (E) 2007.

8. Suppose that only one of the following pairs (x, y) yields the positive integer $\sqrt{x^2 + y^2}$. Then

(A) $x = 25530, y = 29464$; (B) $x = 37615, y = 26855$; (C) $x = 15123, y = 32477$;
(D) $x = 28326, y = 28614$; (E) $x = 22536, y = 27462$.

9. The value of

$$\frac{1}{3+1} + \frac{2}{3^2+1} + \frac{4}{3^4+1} + \frac{8}{3^8+1} + \cdots + \frac{2^{2006}}{3^{2^{2006}}+1}$$

is:

(A) $\frac{1}{2}$; (B) $\frac{1}{2} - \frac{2^{2005}}{3^{2^{2005}} - 1}$; (C) $\frac{1}{2} - \frac{2^{2006}}{3^{2^{2006}} - 1}$; (D) $\frac{1}{2} - \frac{2^{2007}}{3^{2^{2007}} - 1}$; (E) None of the above.

10. Suppose that p and q are prime numbers and they are roots of the equation $x^2 - 99x + m = 0$ for some m. What is the value of $\frac{p}{q} + \frac{q}{p}$?

(A) 9413; (B) $\frac{9413}{194}$; (C) $\frac{9413}{99}$; (D) $\frac{9413}{97}$; (E) None of the above.

11. What is the remainder when $2006 \times 2005 \times 2004 \times 2003$ is divided by 7?

12. If $\dfrac{139}{22} = a + \dfrac{1}{b + \frac{1}{c}}$, where a, b and c are positive integers, find the value of $a + b + c$.

13. Let x be a positive real number. Find the minimum value of $x + \dfrac{1}{x}$.

14. Find the value (in the simplest form) of $\sqrt{45 + 20\sqrt{5}} + \sqrt{45 - 20\sqrt{5}}$.

15. Let n be the number

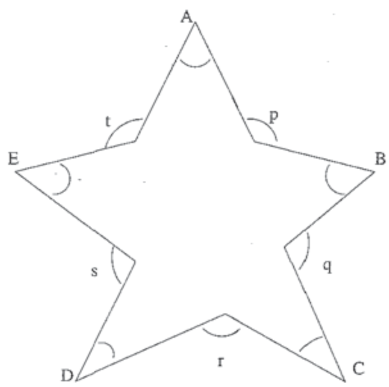
$$(\underbrace{999\,999\,999\;\ldots\;999}_{2006\ 9\text{'s}})^2 - (\underbrace{666\,666\,666\;\ldots\;666}_{2006\ 6\text{'s}})^2.$$

Find the reminder when n is divided by 11.

16. Given that $w > 0$ and that $w - \dfrac{1}{w} = 5$, find the value of $(w + \dfrac{1}{w})^2$.

17. N pieces of candy are made and packed into boxes, with each box containing 45 pieces. If N is a non-zero perfect cube and 45 is one of its factors, what is the least possible number of boxes that can be packed?

18. Consider the following "star" figure.



Given that $\angle p + \angle q + \angle r + \angle s + \angle t = 500°$ and $\angle A + \angle B + \angle C + \angle D + \angle E = x°$, find the value of x.

19. Given that n is a positive integer and $S = 1 + 2 + 3 + \cdots + n$. The units digit of S cannot be some numbers. Find the sum of these numbers.

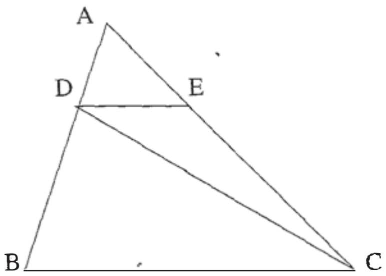
20. Let $m = 76^{2006} - 76$. Find the remainder when m is divided by 100.

21. Let $ABCDEF$ be a hexagon such that the diagonals AD, BE and CF intersect at the point O, and the area of the triangle formed by any three adjacent points is 2 (for example, area of ΔBCD is 2). Find the area of the hexagon.

22. Let C be a circle with radius 2006. Suppose n points are placed inside the circle and the distance between any two points exceed 2006. What is the largest possible n?

23. Let x and y be positive real numbers such that $x^3 + y^3 + \frac{1}{27} = xy$. Find the value of $\frac{1}{x}$.

24. In this question, $S_{\Delta XYZ}$ denotes the area of ΔXYZ. In the following figure, if $DE//BC$, $S_{\Delta ADE} = 1$ and $S_{\Delta ADC} = 4$, find $S_{\Delta DBC}$.



25. What is the product of the real roots of the equation

$$\frac{x^2 + 90x + 2027}{3} = \sqrt{x^2 + 90x + 2055}?$$

26. There are four piles of stones: One with 6 stones, two with 8, and one with 9. Five players numbered 1, 2, 3, 4 and 5 take turns, in the order of their numbers, choosing one of the piles and dividing it into two smaller piles. The loser is the player who cannot do this. State the number of the player who loses.

27. Let $m \neq n$ be two real numbers such that $m^2 = n + 2$ and $n^2 = m + 2$. Find the value of $4mn - m^3 - n^3$.

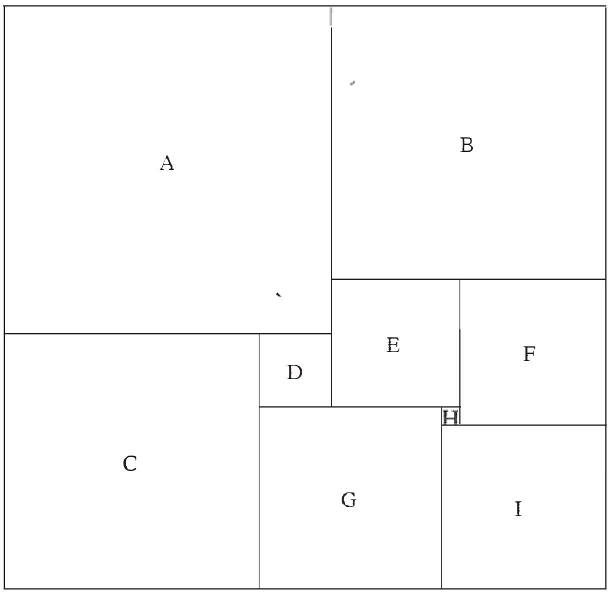
28. There are a few integer values of a such that $\frac{a^2 - 3a - 3}{a - 2}$ is an integer. Find the sum of all these integer values of a.

29. How may pairs of integers (x, y) satisfy the equation

$$\sqrt{x} + \sqrt{y} = \sqrt{200600}?$$

30. The '4' button on my calculator is spoilt, so I cannot enter numbers which contain the digit 4. Moreover, my calculator does not display the digit 4 if 4 is part of an answer either. Thus I cannot enter the calculation 2×14 and do not attempt to do so. Also, the result of multiplying 3 by 18 is displayed as 5 instead of 54 and the result of multiplying 7 by 7 is displayed as 9 instead of 49. If I multiply a positive one-digit number by a positive two-digit number on my calculator and it displays 26, how many possibilities could I have multiplied?

31. The following rectangle is formed by nine pieces of squares of different sizes. Suppose that each side of the square E is of length 7cm. Let the area of the rectangle be x cm^2. Find the value of x.



32. Suppose that n is a positive integer, and a, b are positive real numbers with $a + b = 2$. Find the smallest possible value of

$$\frac{1}{1 + a^n} + \frac{1}{1 + b^n}.$$

33. What is the largest positive integer n for which $n^3 + 2006$ is divisible by $n + 26$?

34. Suppose that the two roots of the equation

$$\frac{1}{x^2-10x-29}+\frac{1}{x^2-10x-45}-\frac{2}{x^2-10x-69}=0$$

are α and β. Find the value of $\alpha+\beta$.

35. Suppose that a, b, x and y are real numbers such that

$$ax+by=3,\quad ax^2+by^2=7,\quad ax^3+by^3=16\quad\text{and}\quad ax^4+by^4=42.$$

Find the value of ax^5+by^5.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Junior Section Solutions)

1. Ans: (A)

   Because 100 is one of the factors.

2. Ans: (B)

   Use $x^2 - 4x + 3 = (x-2)^2 - 1$.

3. Ans: (C)

   Use $1 + 2 + \cdots + 100 = 5050$.

4. Ans: (D)

   The number of diagonals of an n-side polygon is $\dfrac{n(n-3)}{2}$. Hence (D).

5. Ans: (C)

   Because $n^2 < 5^3 = 125$.

6. Ans: (D)

   Let a be the length of AB. Then

$$\frac{\text{area of } \Delta ADE}{\text{area of } \Delta DEC} = \frac{\frac{1}{2}a \cdot a \cdot \sin 60^\circ}{\frac{1}{2}a \cdot a \cdot \sin 30^\circ} = \sqrt{3}.$$

7. Ans: (B)

   Use

$$\frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \ldots + \frac{1}{(x+2005)(x+2006)} =$$

$$\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+2} - \frac{1}{x+3} + \ldots + \frac{1}{x+2005} - \frac{1}{x+2006} = \frac{1}{x+1} - \frac{1}{x+2006} =$$

$$\frac{2005}{(x+1)(x+2006)}$$

8. Ans: (A)

Since $x^2 + y^2$ is a perfect square, its last digit must be 0, 1, 4, 5, 6,or 9. Hence (C) and (D) are not suitable. For (B) and (e), the last two digits are 50 and 60 resp. However, if a number ends with 0, its square must end with two 0's. Thus the ans is (A) where $\sqrt{x^2+y^2}$ turns out to be 38986.

9. Ans: (D)

Since

$$\frac{2^{k+1}}{3^{2^{k+1}}-1} = \frac{2^k}{3^{2^k}-1} - \frac{2^k}{3^{2^k}+1},$$
$$\frac{2^k}{3^{2^k}+1} = \frac{2^k}{3^{2^k}-1} - \frac{2^{k+1}}{3^{2^{k+1}}-1}$$

becomes a telescope sum. Thus the result is $\frac{1}{2} - \frac{2^{2007}}{3^{2^{2007}}-1}$.

10. Ans: (B)

$p + q = 99$. The only possible prime solutions are 2, 97.

11. Ans: 3.

$2006 \times 2005 \times 2004 \times 2003 \equiv 4 \times 3 \times 2 \equiv 3 \pmod 7$.

12. Ans: 16.

Easy calculation shows that $a = 6$, $b = 3$ and $c = 7$.

13. Ans: 2.

$$x + \frac{1}{x} \geq 2\sqrt{x \cdot \frac{1}{x}} = 2,$$

and = holds when $x = 1$.

14. Ans: 10.

$\sqrt{45 + 20\sqrt{5}} + \sqrt{45 - 20\sqrt{5}} = \sqrt{(5 + 2\sqrt{5})^2} + \sqrt{(5 - 2\sqrt{5})^2} = 10.$

15. Ans: 0.

Observe that 11 divides both $\underbrace{999\,999\,999\ \ldots\ 999}_{2006\ 9\text{'s}}$ and $\underbrace{666\,666\,666\ \ldots\ 666}_{2006\ 6\text{'s}}$.

16. Ans: 29.

Use that $(w + \frac{1}{w})^2 = (w - \frac{1}{w})^2 + 4$.

17. Ans: 75.

The least non-zero perfect cube of the form $45m = 3^2 \cdot 5m$ is $3^3 \times 5^3$. Thus the least possible number of boxes that can be packed is $3 \times 5^2 = 75$.

18. Ans: 140.

Use angle sum of polygons,

$$x^\circ + 5 \times 360^\circ - (\angle p + \angle q + \angle r + \angle s + \angle t) = 8 \times 180^\circ,$$

$x = 140$.

19. Ans: 22.

$S = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. The units digit of S can only be $0, 1, 3, 5, 6, 8$. Thus S cannot end with $2, 4, 7$ or 9. The sum is 22.

20. Ans: 0.

$m = 76 \cdot (76^{2005} - 1) = 76 \times 75 \times k$ for some k. Thus 100 divides m.

21. Ans: 12.

Since area of ΔABC = area of ΔFAB, $FC//AB$. Similarly $FC//ED$.

By the same reason, $FE//AD//BC$ and $AF//BE//CD$. Thus $OFDE$, $ODBC$ and $OBAF$ are all parallelogram and each of them has area 4. Thus the total area of the hexagon is 12.

22. Ans: 5.

Clearly, none of the points is in the centre O. Label the points $A_1, A_2, \ldots A_n$ in clockwise direction. Note that no two points lie on the same radius. Now, for any two points A_i, A_{i+1} ($A_{n+1} = A_0$), the angle $\angle A_iOA_{i+1} > 60$. Therefore, $n < 6$. For $n = 5$, just take the points to be the vertices of a suitable regular pentagon, then it will satisfy our condition. Therefore, $n = 5$.

23. Ans: 3.

Using the fact that AM$\geq$ GM, we get $x^3 + y^3 + 1/27 \geq 3\sqrt[3]{x^3y^3(1/27)} = xy$ and = holds only when $x^3 = 1/27$. Thus $x = 1/3$.

24. Ans: 12.

$S_{\triangle ADE} : S_{\triangle CDE} = 1 : 3$. Hence $AE : EC = AD : DB = 1 : 3$. Thus $AE : AC = AD : AB = 1 : 4$. Hence $S_{\triangle ABC} = 16$ and $S_{\triangle DBC} = 12$.

25. Ans: 2006.

Let $u = \sqrt{x^2 + 90x + 2055}$. Then $u^2 - 28 = 3u$. Solve to find the positive root $u = 7$. Hence $x^2 + 90x + 2006 = 0$, so the product of the real roots is 2006.

26. Ans: 3.

Observe that we begin with 4 piles of stones and end up with $6+8+8+9 = 31$ piles of (one) stones. At the end of each turn, there is exactly one more pile of stones than the beginning of the turn. Thus, there can be exactly $31 - 4 = 27$ legal turns. Hence, Player 3 is the first player who cannot make a move.

27. Ans: 0.

Since $m \neq n$ and $m^2 - n^2 = (n - m)$, we get $m + n = -1$. Thus $m^2 + n^2 = (m + n) + 4 = 3$ and $mn = \frac{1}{2}[(m + n)^2 - m^2 - n^2] = -1$. Thus $4mn - m^3 - n^3 = 4mn - (m + n)(m^2 + n^2 - mn) = -4 - (-1)(3 - (-1)) = 0$.

28. Ans: 8.

Since $\dfrac{a^2 - 3a - 3}{a - 2} = a - 1 - \dfrac{5}{a - 2}$, thus $a = -3, 1, 3, 7$. Their sum is 8.

29. Ans: 11.

Since $x = 200600 + y - 20\sqrt{2006y}$ is an integer, $y = 2006u^2$ for some positive integer u. Similarly $x = 2006v^2$ for some positive integer v. Thus $u + v = 10$. There are 11 pairs in total, namely $(0, 10), (1, 9), \ldots, (10, 0)$.

30. Ans: 6.

Since the product is at most a three-digit number, the possible answers I should get is 26, 426, 246 or 264. Since

$$\begin{aligned}
26 &= \mathbf{1} \times \mathbf{26} = \mathbf{2} \times \mathbf{13} \\
426 &= 1 \times 426 = 2 \times 213 = 3 \times 142 = \mathbf{6} \times \mathbf{71} \\
246 &= 1 \times 246 = 2 \times 123 = \mathbf{3} \times \mathbf{82} = 6 \times 41 \\
264 &= 1 \times 264 = 2 \times 132 = \mathbf{3} \times \mathbf{88} = 4 \times 66 \\
&= 6 \times 44 = \mathbf{8} \times \mathbf{33} = 11 \times 24 = 12 \times 22,
\end{aligned}$$

there are 6 possibilities.

31. Ans: 1056.

Let us use $|X|$ to denote the length of one side of square X. Thus $|E| = 7$. Let $|H|$ be a and $|D|$ be b. Then $|F|$ is $a + 7$, $|B|$ is $a + 14$, $|I|$ is $2a + 7$, $|G|$ is $3a + 7$, $|C|$ is $3a + b + 7$, and $|A|$ is $3a + 2b + 7$. Hence we have

$$(3a + 2b + 7) + (a + 14) = (3a + b + 7) + (3a + 7) + (2a + 7)$$

$$(3a + 2b + 7) + (3a + b + 7) = (a + 14) + (a + 7) + (2a + 7).$$

Therefore $a = 1$ and $b = 4$. Hence the area of the rectangle is $32 \times 33 = 1056$.

32. Ans: 1.

Note that $ab \leq 1$ and $(ab)^n \leq 1$.

$$\frac{1}{1 + a^n} + \frac{1}{1 + b^n} = \frac{1 + a^n + b^n + 1}{1 + a^n + b^n + (ab)^n} \geq 1.$$

When $a = b = 1$, we get 1. Thus, the smallest value is 1.

33. Ans: 15544.

$n^3 + 2006 = (n + 26)(n^2 - 26n + 676) - 15570$. So if $n + 26$ divides $n^3 + 2006$, $n + 26$ must divide 15570. Thus the largest n is 15544.

34. Ans: 10.

Let $y = x^2 - 10x - 29$. Then $x^2 - 10x - 45 = y - 16$, $x^2 - 10x - 69 = y - 40$. Thus $\frac{1}{y} + \frac{1}{y - 16} - \frac{2}{y - 40} = 0$. Therefore $y = 10$. So $y = x^2 - 10x - 29 = 10$. Hence $x^2 - 10x - 39 = 0$. Therefore $\alpha + \beta = 10$.

35. Ans: 20.

$ax^3 + by^3 = (ax^2 + by^2)(x + y) - (ax + by)xy$. Thus, we get $16 = 7(x + y) - 3xy$. Similarly, $ax^4 + by^4 = (ax^3 + by^3)(x + y) - (ax^2 + by^2)xy$. So, we get $42 = 16(x + y) - 7xy$. Solving, we get $x + y = -14$ and $xy = -38$. Therefore,

$$ax^5 + by^5 = (ax^4 + by^4)(x + y) - (ax^3 + by^3)xy = 42(-14) - 16(-38) = 20.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Junior Section, Special Round)

Saturday, 24 June 2006 — 0930– 1230

Important:

Attempt as many questions as you can.

No calculators are allowed.

Show all the steps in your working.

Each question carries 10 marks.

1. Find all integers x, y that satisfy the equation

$$x + y = x^2 - xy + y^2.$$

2. The fraction $\frac{2}{3}$ can be expressed as a sum of two distinct unit fractions: $\frac{1}{2} + \frac{1}{6}$. Show that the fraction $\frac{p-1}{p}$, where $p \geq 5$ is a prime, cannot be expressed as a sum of two distinct unit fractions.

3. Suppose that each of n people knows exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything he knows, while B tells A nothing. What is the minimum of phone calls between pairs of people needed for everyone to know everything?

4. In $\triangle ABC$, the bisector of $\angle B$ meets AC at D and the bisector of $\angle C$ meets AB at E. These bisectors intersect at O and $OD = OE$. If $AD \neq AE$, prove that $\angle A = 60°$.

5. You have a large number of congruent equilateral triangular tiles on a table and you want to fit n of them together to make a convex equiangular hexagon (i.e., one whose interior angles are $120°$). Obviously, n cannot be any positive integer. The first three feasible n are 6, 10 and 13. Show that 12 is not feasible but 14 is.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Junior Section, Special Round Solutions)

1. Solving for y, we get:

$$y = \frac{x+1 \pm \sqrt{-3(x-1)^2+4}}{2}.$$

Thus $3(x-1)^2 \leq 4$, i.e.,

$$1 - \frac{2}{\sqrt{3}} \leq x \leq 1 + \frac{2}{\sqrt{3}}.$$

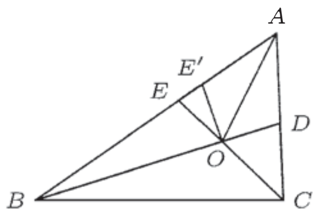
Thus $x = 0, 1, 2$ and $(x, y) = (0,0), (0,1), (1,0), (1,2), (2,1), (2,2)$ are all the solutions.

2. Note that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Since $\frac{1}{2}$ and $\frac{1}{3}$ are the two largest unit fractions and $\frac{p-1}{p} > \frac{5}{6}$ for all $p \geq 7$, the result is true for $p \geq 7$. Suppose $\frac{4}{5} = \frac{1}{a} + \frac{1}{b}$, with $a > b$. Then $\frac{4}{5} = \frac{1}{a} + \frac{1}{b} < \frac{2}{b}$. Therefore $2b < 5$, i.e., $b = 2$ and there is no solution for a.

3. We claim that the minimum of calls needed is $2n-2$. Let A be a particular person, the $2n-2$ calls made by A to each of the persons and vice versa will leave everybody informed. Thus at most $2n-2$ calls are needed.

Next we prove that we need at least $2n-2$ calls. Suppose that there is a sequence of calls that leaves everybody informed. Let B be the first person to be fully informed and that he receives his last piece of information at the pth call. Then each of the remaining $n-1$ people must have placed at least one call prior to p so that B can be fully informed. Also these people must received at least one call after p since they were still not fully informed at the pth call. Thus we need at least $2(n-1)$ calls.

4.

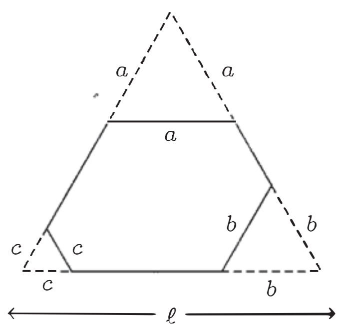


Assume that $AE > AD$. Let $\angle A = 2a$, $\angle B = 2b$, $\angle C = 2c$ and $\angle ADO = x$. Now AO bisects $\angle A$. Let E' be the point on AB such that $OE' = OE$. Since $AE > AD$, E' lies between A and E. We have $\triangle AE'O \equiv \triangle ADO$ (SAS). Thus $OE' = OD = OE$ and $x = \angle ADO = \angle AE'O = \angle BEO$. From $\triangle ABD$ and $\triangle BEC$, we have $2a + x + b = 180°$ and $x + 2b + c = 180°$. Thus $2a = b + c$ and so $\angle BAC = 2a = 60°$.

5. Assume that the tiles are of side length 1. Note the number of tiles required to form an equilateral triangle of length x is $1 + 3 + \cdots + (2x-1) = x^2$. The triangle formed by extending the alternate sides of the hexagon must be an equilateral triangle of side length say ℓ. The hexagon is formed by removing the three corner equilateral triangles of side lengths a, b, c and $\ell > a+b, a+c, b+c$. An equilateral triangle of side length x contains x^2 tiles. Thus n is feasible if and only if

$$n = \ell^2 - a^2 - b^2 - c^2 \quad \text{and} \quad \ell > a+b,\ a+c,\ b+c$$

Take $\ell = 5$, $a = 3$, $b = c = 1$. Then $n = 14$ and so is feasible.

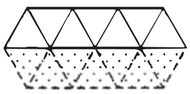


Next we show that $n = 12$ is not feasible. Let $a \geq b \geq c$. For fixed ℓ, we want to find a lower bound for n. For this purpose, we may assume that $a + b = \ell - 1$. Thus $b \leq (\ell - 1)/2$. If $a = \quad - 1 - k$, then $b, c = k \leq (\ell-1)/2$. Thus

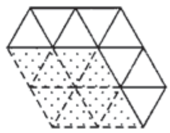
$$n \geq \ell^2 - 2k^2 - (\ell - 1 - k)^2 = (2\ell - 1) - (3k^2 - 2k(\ell - 1)).$$

Since $1 \leq k \leq (\ell - 1)/2$, we see that the maximum value of $3k^2 - 2k($
either $k = 1$ or $k = ($ $\qquad$ $n \geq 4\ell - 6 = A$ (when $k = 1$) and $n \geq (\ell^2 + 6\ell - 3)/4 = B$ when $k = (\ell-1)/2$. Thus $n \geq 6$ for $\ell = 3$, $n \geq 10$ for $\ell = 4$, $n \geq 13$ for $\ell = 5$, $n \geq 18$ for $\ell = 6$. For $\ell \geq 7$, $n \geq 22$. Thus we only have to check the case $\ell =$
$\ell = 3$, we have $a = b = c = 1$. For $\ell = 4$, we have $(a, b, c) = (2, 1, 1), (1, 1, 1)$. These give $n = 6, 10, 13$. Thus 12 is not feasible.

(Note: You can show that 14 is feasible by drawing a hexagon with 14 tiles. It is possible to show that 12 is not feasible by brute force. One of the sides must be at least of length 2. If one side has length 3, we need at least 14 tiles. In Fig. 1, the top side is of length 3 and the 7 tiles in the unshaded region must be present. No matter what you do, the 7 tiles in the shaded region must also be present. In fact this is the smallest hexagon with one side of length 3. If two adjacent sides are of length 2, then we need at least 16 tiles (Fig 2). If three consecutive sides are of lengths 2, 1, 2, then we need at least 13 tiles (Fig 3). The only other case is 2, 1, 1, 2, 1, 1 which gives 10 tiles (Fig 4). Thus 12 is not feasible.)

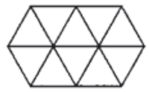


(1)



(2)

(3)



(4)

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Senior Section)

Tuesday, 30 May 2006 0930-1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answers in the answer sheet by shading the bubbles containing the letters (A, B, C, D *or* E) *corresponding to the correct answers.*

For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

1. Let $p = 2^{3009}$, $q = 3^{2006}$ and $r = 5^{1003}$. Which of the following statements is true?

   (A) $p < q < r$ (B) $p < r < q$ (C) $q < p < r$ (D) $r < p < q$

   (E) $q < r < p$

2. Which of the following numbers is the largest?

   (A) 30^{20} (B) 10^{30} (C) $30^{10} + 20^{20}$ (D) $(30 + 10)^{20}$ (E) $(30 \times 20)^{10}$

3. What is the last digit of the number

$$2^2 + 20^{20} + 200^{200} + 2006^{2006}?$$

   (A) 2 (B) 4 (C) 6 (D) 8 (E) 0

4. Let x be a number such that $x + \dfrac{1}{x} = 4$. Find the value of $x^3 + \dfrac{1}{x^3}$.

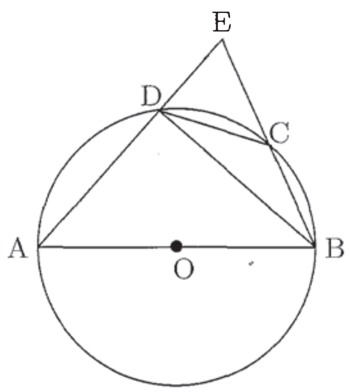
   (A) 48 (B) 50 (C) 52 (D) 54 (E) None of the above

5. Consider the two curves $y = 2x^3 + 6x + 1$ and $y = -\dfrac{3}{x^2}$ in the Cartesian plane. Find the number of distinct points at which these two curves intersect.

(A) 1 (B) 2 (C) 3 (D) 0 (E) 5

6. In the following figure, AB is the diameter of a circle with centre at O. It is given that $AB = 4$ cm, $BC = 3$ cm, $\angle ABD = \angle DBE$. Suppose the area of the quadrilateral $ABCD$ is x cm^2 and the area of $\triangle DCE$ is y cm^2. Find the value of the ratio $\dfrac{x}{y}$.

(A) 7 (B) 8 (C) 4 (D) 5 (E) 6



7. Five students A, B, C, D and E form a team to take part in a 5-leg relay competition. If A cannot run the first leg and D cannot run the last leg, how many ways can we arrange them to run the relay?

(A) 74 (B) 76 (C) 78 (D) 80 (E) 82

8. There are n balls in a box, and the balls are numbered $1, 2, 3, \ldots, n$ respectively. One of the balls is removed from the box, and it turns out that the sum of the numbers on the remaining balls in the box is 5048. If the number on the ball removed from the box is m, find the value of m.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

9. Suppose a, b, c are real numbers such that $a + b + c = 0$ and $abc = -100$. Let $x = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Which of the following statements is true?

(A) $x > 0$ (B) $x = 0$ (C) $-1 < x < 0$ (D) $-100 < x < -1$

(E) $x < -100$

10. Let a and b be positive real numbers such that

$$\frac{1}{a} - \frac{1}{b} - \frac{1}{a+b} = 0.$$

Find the value of $\left(\frac{b}{a} + \frac{a}{b}\right)^2$.

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

11. Find the value of

$$\frac{2006^2 - 1994^2}{1600}.$$

12. Find the smallest natural number n which satisfies the inequality

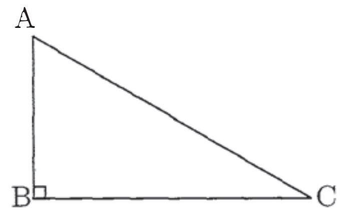
$$2006^{1003} < n^{2006}.$$

13. Find the smallest integer greater than $(1 + \sqrt{2})^3$.

14. Find the number of pairs of positive integers (x, y) which satisfy the equation

$$20x + 6y = 2006.$$

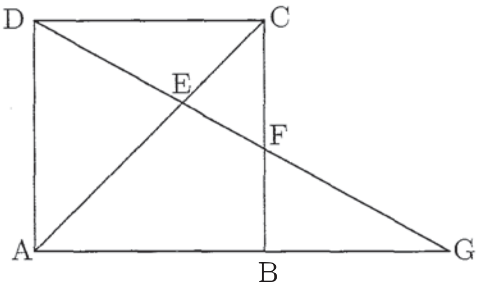
15. $\triangle ABC$ is a right-angled triangle with $\angle ABC = 90°$. A circle C_1 is drawn with AB as diameter, and another circle C_2 is drawn with BC as diameter. The circles C_1 and C_2 meet at the points B and P. If $AB = 5$ cm, $BC = 12$ cm and $BP = x$ cm, find the value of $\frac{2400}{x}$.



16. Evaluate

$$\frac{1}{\log_2 12\sqrt{5}} + \frac{1}{\log_3 12\sqrt{5}} + \frac{1}{\log_4 12\sqrt{5}} + \frac{1}{\log_5 12\sqrt{5}} + \frac{1}{\log_6 12\sqrt{5}}.$$

17. In the diagram below, $ABCD$ is a square. The points A, B and G are collinear. The line segments AC and DG intersect at E, and the line segments DG and BC intersect at F. Suppose that $DE = 15$ cm, $EF = 9$ cm, and $FG = x$ cm. Find the value of x.



18. Find the sum of the coefficients of the polynomial

$$(4x^2 - 4x + 3)^4(4 + 3x - 3x^2)^2.$$

19. Different positive 3-digit integers are formed from the five digits 1, 2, 3, 5, 7, and repetitions of the digits are allowed. As an example, such positive 3-digit integers include 352, 577, 111, etc. Find the sum of all the distinct positive 3-digit integers formed in this way.

20. Find the value of $\dfrac{1}{\sin 10^\circ} - 4\sin 70^\circ$.

21. Let $w = 1 + \sqrt[5]{2} + \sqrt[5]{4} + \sqrt[5]{8} + \sqrt[5]{16}$. Find the value of $(1 + \frac{1}{w})^{30}$.

22. Suppose A and B are two angles such that

$$\sin A + \sin B = 1 \quad \text{and} \quad \cos A + \cos B = 0.$$

Find the value of $12\cos 2A + 4\cos 2B$.

23. Consider the 800-digit integer

$$234523452345\cdots2345.$$

The first m digits and the last n digits of the above integer are crossed out so that the sum of the remaining digits is 2345. Find the value of $m+n$.

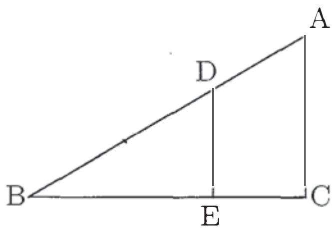
24. Let a and b be two integers. Suppose that $\sqrt{7-4\sqrt{3}}$ is a root of the equation $x^2+ax+b=0$. Find the value of $b-a$.

25. Suppose x and y are integers such that

$$(x-2004)(x-2006)=2^y.$$

Find the largest possible value of $x+y$.

26. In the following diagram, $\angle ACB = 90°$, $DE \perp BC$, $BE = AC$, $BD = \frac{1}{2}$ cm, and $DE + BC = 1$ cm. Suppose $\angle ABC = x°$. Find the value of x.



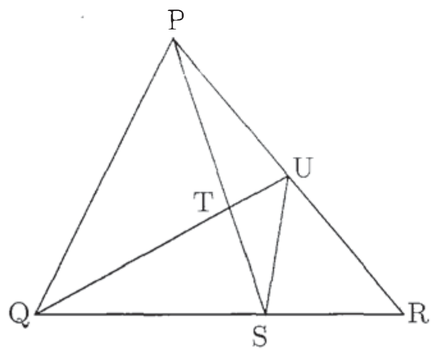
27. If

$$f(x) = \frac{1}{\sqrt[3]{x^2+2x+1}+\sqrt[3]{x^2-1}+\sqrt[3]{x^2-2x+1}}$$

for all positive integers x, find the value of

$$f(1)+f(3)+f(5)+\cdots+f(997)+f(999).$$

28. In the figure below, S is a point on QR and U is a point on PR. The line segments PS and QU intersect at the point T. It is given that $PT=TS$ and $QS=2RS$. If the area of $\triangle PQR$ is 150 cm^2 and the area of $\triangle PSU$ is x cm^2. Find the value of x.



29. Let a and b be two integers. Suppose $x^2 - x - 1$ is a factor of the polynomial $ax^5 + bx^4 + 1$. Find the value of a.

30. If $\sin\theta - \cos\theta = \dfrac{\sqrt{6}-\sqrt{2}}{2}$, find the value of $24(\sin^3\theta - \cos^3\theta)^2$.

31. How many ordered pairs of positive integers (x, y) satisfy the equation

$$x\sqrt{y} + y\sqrt{x} + \sqrt{2006xy} - \sqrt{2006x} - \sqrt{2006y} - 2006 = 0?$$

32. Find the remainder when the integer

$$1 \times 3 \times 5 \times 7 \times \cdots \times 2003 \times 2005$$

is divided by 1000.

33. Let $f : \mathbb{N} \to \mathbb{Q}$ be a function, where $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{Q}$ denotes the set of rational numbers. Suppose that $f(1) = \dfrac{3}{2}$, and

$$f(x+y) = \left(1 + \frac{y}{x+1}\right) f(x) + \left(1 + \frac{x}{y+1}\right) f(y) + x^2y + xy + xy^2$$

for all natural numbers x, y. Find the value of $f(20)$.

34. Suppose $x_0, x_1, x_2, \cdots$ is a sequence of numbers such that $x_0 = 1000$, and

$$x_n = -\frac{1000}{n}(x_0 + x_1 + x_2 + \cdots + x_{n-1})$$

for all $n \geq 1$. Find the value of

$$\frac{1}{2^2}x_0 + \frac{1}{2}x_1 + x_2 + 2x_3 + 2^2x_4 + \cdots + 2^{997}x_{999} + 2^{998}x_{1000}.$$

35. Let p be an integer such that both roots of the equation

$$5x^2 - 5px + (66p - 1) = 0$$

are positive integers. Find the value of p.

Singapore Mathematical Society

Singapore Mathematical Olympiad 2006

(Senior Section Solutions)

1. Ans: D

$$\begin{aligned} p &= 2^{3009} = 2^{3\times 1003} = (2^3)^{1003} = 8^{1003}. \\ q &= 3^{2006} = 3^{2\times 1003} = (3^2)^{1003} = 9^{1003}. \\ r &= 5^{1003}. \end{aligned}$$

Thus, we have $r < p < q$.

2. Ans: D

$$\begin{aligned} &30^{20} = (30^2)^{10} = 900^{10}. \\ &10^{30} = (10^3)^{10} = 1000^{10}. \\ &30^{10} + 20^{20} = 30^{10} + (20^2)^{10} = 30^{10} + 400^{10} < 2 \cdot 400^{10}. \\ &\qquad < (2 \times 400)^{10} = 800^{10}. \\ &(30+10)^{20} = 40^{20} = (40^2)^{10} = 1600^{10}. \\ &(30 \times 20)^{10} = 600^{10}. \end{aligned}$$

Therefore, the largest number is $(30+10)^{20} = 1600^{10}$.

3. Ans: E

First we observe that $2^2 = 4$. Since 20^{20} is divisible by 10, it follows that its last digit is 0. Similarly, the last digit of 200^{200} is 0. The last digit of 2006^{2006} is the same as that of 6^{2006}. Since $6 \times 6 = 36$, it follows that the last digit of any positive integral power of 6 is 6. Thus the last digit of 2006^{2006} is 6. Now, $4+0+0+6 = 10$. Thus the last digit of $2^2 + 20^{20} + 200^{200} + 2006^{2006}$ is 0.

4. Ans: C

$$\begin{aligned} x + \frac{1}{x} = 4 &\Longrightarrow (x + \frac{1}{x})^3 = 4^3 = 64 \\ &\Longrightarrow x^3 + 3x + 3 \cdot \frac{1}{x} + \frac{1}{x^3} = 64 \\ &\Longrightarrow x^3 + \frac{1}{x^3} + 3(x + \frac{1}{x}) = 64 \\ &\Longrightarrow x^3 + \frac{1}{x^3} = 64 - 3(4) = 52. \end{aligned}$$

5. Ans: A

To find the intersection points of the two curves, we solve their equations simultaneously.

$$\begin{cases} y = 2x^3 + 6x + 1, \\ y = -\dfrac{3}{x^2}. \end{cases}$$

Thus, we have

$$\begin{aligned} 2x^3 + 6x + 1 = -\frac{3}{x^2} &\implies 2x^5 + 6x^3 + x^2 + 3 = 0 \\ &\implies (2x^3 + 1)(x^2 + 3) = 0 \\ &\implies 2x^3 + 1 = 0, \ \text{ since } \ x^2 + 3 > 0 \\ &\implies x^3 = -\frac{1}{2} \\ &\implies x = -\frac{1}{\sqrt[3]{2}} \\ &\implies y = -\frac{3}{\sqrt[3]{4}}. \end{aligned}$$

Thus, the only point of intersection is $\left(-\frac{1}{\sqrt[3]{2}}, -\frac{3}{\sqrt[3]{4}}\right)$.

6. Ans: A

Observe that $\triangle ABD \cong \triangle EBD$. Thus, $BE = AB = 4$, $AD = DE$. Hence

$$\text{Area of } \triangle ACE = 2 \times \text{ Area of } \triangle DCE.$$

Since $BC = 3$, $EC = EB - BC = 4 - 3 = 1$. Thus,

$$\begin{aligned} \text{Area of } \triangle ABC &= 3 \times \text{ Area of } \triangle ACE. \\ &= 6 \times \text{ Area of } \triangle DCE. \end{aligned}$$

Thus,

$$\begin{aligned} \text{Area of } ABCD &= \text{Area of } \triangle ABC + \text{ Area of } \triangle ACD \\ &= 6 \times \text{ Area of } \triangle DCE + \text{ Area of } DCE \\ &= 7 \times \text{ Area of } \triangle DCE. \end{aligned}$$

7. Ans: C

Total number of ways is

$$5! - 4! - 4! + 3! = 120 - 24 - 24 + 6 = 78.$$

Alternatively, we consider the following cases:

Case (i): A and D do not run the first or last leg. In this case, the number of arrangements is $3 \times 2 \times 3! = 36$.

Case (ii): D runs the first leg. In this case, number of arrangements is $4! = 24$.

Case (iii): D does not run the first leg and A runs the last leg. In this case, number of ways is $3 \times 3! = 18$.

Therefore, total number of ways $= 36 + 24 + 18 = 78$.

8. Ans: B

Let m be the number of the removed ball. Then we have $1 \leq m \leq n$, and

$$\begin{aligned}
&\Longrightarrow \quad 1 + 2 + 3 + \cdots + n - m = 5048 \\
&\Longrightarrow \frac{n(n+1)}{2} - m = 5048 \\
&\Longrightarrow 1 \leq \frac{n(n+1)}{2} - 5048 \leq n \\
&\Longrightarrow \frac{(n-1)n}{2} \leq 5048 \text{ and } \frac{n(n+1)}{2} \geq 5049.
\end{aligned}$$

Observe that $\frac{99 \times 100}{2} = 4950$, $\frac{100 \times 101}{2} = 5050$ and $\frac{101 \times 102}{2} = 5151$. It follows that $n = 100$. Hence $m = \frac{100 \times 101}{2} - 5048 = 2$.

9. Ans: A

Observe that

$$0 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca).$$

Since $abc = -100$, it follows that $a \neq 0$, $b \neq 0$ and $c \neq 0$. Thus, $a^2 + b^2 + c^2 > 0$, and it follows that $ab + bc + ca < 0$. Therefore,

$$x = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ca}{abc} = \frac{ab + bc + ca}{-100} > 0.$$

10. Ans: B

Let $x = \dfrac{b}{a}$. Then $b = ax$. Hence we have

$$\begin{aligned}\frac{1}{a}-\frac{1}{b}-\frac{1}{a+b}=0 &\Longrightarrow \frac{1}{a}-\frac{1}{ax}-\frac{1}{a+ax}=0\\ &\Longrightarrow \frac{1}{a}\left(1-\frac{1}{x}-\frac{1}{1+x}\right)=0\\ &\Longrightarrow x(x+1)-(x+1)-x=0\\ &\Longrightarrow x^2-x-1=0\\ &\Longrightarrow x=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}.\end{aligned}$$

Since a and b are positive, it follows that $x > 0$, and thus $x = \dfrac{1+\sqrt{5}}{2}$. Then

$$\begin{aligned}\left(\frac{b}{a}+\frac{a}{b}\right)^2=\left(x+\frac{1}{x}\right)^2 &= \left(\frac{1+\sqrt{5}}{2}+\frac{2}{1+\sqrt{5}}\right)^2\\ &= \left(\frac{1+\sqrt{5}}{2}+\frac{2}{1+\sqrt{5}}\cdot\frac{1-\sqrt{5}}{1-\sqrt{5}}\right)^2\\ &= \left(\frac{1+\sqrt{5}}{2}+\frac{2-2\sqrt{5}}{-4}\right)^2\\ &= \left(\frac{1+\sqrt{5}}{2}-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^2=(\sqrt{5})^2=5.\end{aligned}$$

11. Ans: 30

$$\begin{aligned}\frac{2006^2-1994^2}{1600} &= \frac{(2006+1994)\times(2006-1994)}{1600}\\ &= \frac{4000\times 12}{1600}\\ &= 30.\end{aligned}$$

12. Ans: 45

$$2006^{1003} < n^{2006} \Longleftrightarrow 2006 < n^2.$$

Since $44^2 = 1936 < 2006$ and $45^2 = 2025 > 2006$, it follows that the smallest natural number n satisfying $2006^{1003} < n^{2006}$ is 45.

13. Ans: 15

$$(1+\sqrt{2})^3 = 1 + 3\sqrt{2} + 3 \times 2 + 2\sqrt{2} = 7 + 5\sqrt{2}.$$

Observe that $1.4 < \sqrt{2} < 1.5$. Thus

$$\begin{aligned} & 7 + 5 \times 1.4 < 7 + 5\sqrt{2} < 7 + 5 \times 1.5 \\ \Longrightarrow \quad & 14 < 7 + 5\sqrt{2} < 14.5. \end{aligned}$$

Therefore, the smallest integer greater than $(1+\sqrt{2})^3$ is 15.

14. Ans: 34

$$20x + 6y = 2006 \Longleftrightarrow 10x + 3y = 1003.$$

The solutions are

$$(1, 331), (4, 321), (7, 311), \cdots, (94, 21), (97, 11), (100, 1).$$

Thus there are 34 solutions.

15. Ans: 520

Drop the perpendicular from B to AC meeting AC at Q. Then $\angle AQB = 90°$, and thus Q lies on C_1. Similarly, Q lies on C_2. Thus, $Q = P$. Now, $\triangle ABP \sim \triangle ACB$. Thus

$$\frac{BP}{AB} = \frac{BC}{AC} \Longrightarrow \frac{x}{5} = \frac{12}{\sqrt{12^2 + 5^2}} \Longrightarrow x = \frac{60}{13}.$$

Therefore,

$$\frac{2400}{x} = 2400 \times \frac{13}{60} = 520.$$

16. Ans: 2

$$\begin{aligned} & \frac{1}{\log_2 12\sqrt{5}} + \frac{1}{\log_3 12\sqrt{5}} + \frac{1}{\log_4 12\sqrt{5}} + \frac{1}{\log_5 12\sqrt{5}} + \frac{1}{\log_6 12\sqrt{5}} \\ &= \log_{12\sqrt{5}} 2 + \log_{12\sqrt{5}} 3 + \log_{12\sqrt{5}} 4 + \log_{12\sqrt{5}} 5 + \log_{12\sqrt{5}} 6 \\ &= \log_{12\sqrt{5}} 2 \times 3 \times 4 \times 5 \times 6 \\ &= \log_{12\sqrt{5}} 720 \\ &= 2, \text{ since } (12\sqrt{5})^2 = 720. \end{aligned}$$

17. Ans: 16

Let $AB = y$ cm. Since $\triangle DCF \sim \triangle GBF$, we have

$$\frac{DC}{DF} = \frac{GB}{GF} \Longrightarrow \frac{y}{24} = \frac{GB}{x} \Longrightarrow GB = \frac{xy}{24}.$$

Since $\triangle DCE \sim \triangle GAE$, we have

$$\frac{DC}{DE} = \frac{GA}{GE} \Longrightarrow \frac{y}{15} = \frac{y + GB}{9 + x} \Longrightarrow GB = \frac{y(9+x)}{15} - y.$$

Hence we have

$$\frac{xy}{24} = \frac{y(9+x)}{15} - y \Longrightarrow \frac{x}{24} = \frac{9+x}{15} - 1 \Longrightarrow x = 16.$$

18. Ans: 1296

Let

$$(4x^2 - 4x + 3)^4(4 + 3x - 3x^2)^2 = A_{12}x^{12} + \cdots + A_1x + A_0.$$

Substitute $x = 1$ to obtain

$$\begin{aligned} A_{12} + A_{11} + \cdots + A_1 + A_0 &= (0 - 0 + 3)^4(4 + 0 - 0)^2 \\ &= 1296. \end{aligned}$$

19. Ans: 49950

Number of three-digit integers formed $= 5^3 = 125$. Observe that each of the five digits $1, 2, 3, 5, 7$ appears 25 times in the first, second and third digits of the integers formed. Thus,

$$\begin{aligned} \text{sum} &= 25 \times (1 + 2 + 3 + 5 + 7) \times 100 \\ &\quad + 25 \times (1 + 2 + 3 + 5 + 7) \times 10 \\ &\quad + 25 \times (1 + 2 + 3 + 5 + 7) \\ &= 49950. \end{aligned}$$

20. Ans: 2

$$\begin{aligned} \frac{1}{\sin 10^\circ} - 4\sin 70^\circ &= \frac{1 - 4\sin 70^\circ \sin 10^\circ}{\sin 10^\circ} \\ &= \frac{1 - 2(\cos 60^\circ - \cos 80^\circ)}{\sin 10^\circ} \\ &= \frac{1 - 2(\frac{1}{2} - \cos 80^\circ)}{\sin 10^\circ} \\ &= \frac{2\cos 80^\circ}{\sin 10^\circ} \\ &= \frac{2\sin 10^\circ}{\sin 10^\circ} \\ &= 2. \end{aligned}$$

Here we have used the formula:

$$\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}.$$

21. Ans: 64

Let $y = \sqrt[5]{2}$. Then $y^5 = 2$. Therefore,

$$\begin{aligned}
\left(1 + \frac{1}{w}\right)^{30} &= \left(1 + \frac{1}{1 + y + y^2 + y^3 + y^4}\right)^{30} \\
&= \left(1 + \frac{y-1}{y^5 - 1}\right)^{30} \\
&= \left(1 + \frac{y-1}{2-1}\right)^{30} \\
&= y^{30} \\
&= (y^5)^6 = 2^6 = 64.
\end{aligned}$$

22. Ans: 8

$$\sin A + \sin B = 1 \Longrightarrow \sin A = 1 - \sin B.$$
$$\cos A + \cos B = 0 \Longrightarrow \cos A = -\cos B.$$

Thus,

$$\begin{aligned}
\cos^2 A + \sin^2 A = 1 &\Longrightarrow (-\cos B)^2 + (1 - \sin B)^2 = 1 \\
&\Longrightarrow \cos^2 B + 1 - 2\sin B + \sin^2 B = 1 \\
&\Longrightarrow \sin B = \frac{1}{2}.
\end{aligned}$$

Thus, $\sin A = 1 - \frac{1}{2} = \frac{1}{2}$. Therefore,

$$\begin{aligned}
12\cos 2A + 4\cos 2B &= 12(1 - 2\sin^2 A) + 4(1 - 2\sin^2 B) \\
&= 12\left(1 - 2\left(\frac{1}{2}\right)^2\right) + 4\left(1 - 2\left(\frac{1}{2}\right)^2\right) \\
&= 8.
\end{aligned}$$

23. Ans: 130

Note that $2+3+4+5=14$. Thus the sum of the 800 digits is $200\times 14=2800$. Thus we need to cross out digits with a sum equal to $2800-2345=455$.

Observe that $455=32\times 14+7$. Thus we have to cross out 32 blocks of four digits '2345' either from the front or the back, a '2' from the front that remains and a '5' from the back that remains. Thus, $m+n=32\times 4+2=130$.

24. Ans: 5

First we simplify $\sqrt{7-4\sqrt{3}}$, and write

$$\sqrt{7-4\sqrt{3}}=x+y\sqrt{3},$$

where x, y are rational numbers. Then

$$7-4\sqrt{3}=(x+y\sqrt{3})^2=x^2+3y^2+2xy\sqrt{3}.$$

Thus, we have

$$\begin{cases} x^2+3y^2 & = 7, \\ 2xy & = \text{-4}. \end{cases}$$

Substituting the second equation into the first one, we get

$$\begin{aligned} x^2+3\left(-\frac{2}{x}\right)^2=7 &\Longrightarrow x^4-7x^2+12=0 \\ &\Longrightarrow (x^2-4)(x^2-3)=0 \\ &\Longrightarrow x=2,-2,\sqrt{3},-\sqrt{3}. \end{aligned}$$

Since x is a rational number, $x\neq\sqrt{3}$ and $x\neq-\sqrt{3}$. When $x=2$, $y=\dfrac{-4}{2\times 2}=-1$. When $x=-2$, $y=1$. Thus, $\sqrt{7-4\sqrt{3}}=2-\sqrt{3}$ or $-2+\sqrt{3}$. Since $\sqrt{7-4\sqrt{3}}>0$, it follows that $\sqrt{7-4\sqrt{3}}=2-\sqrt{3}$. Thus the 2nd root of the equation is $2+\sqrt{3}$, since the coefficients of the equation are integers. Therefore, $a=-(2-\sqrt{3})-(2+\sqrt{3})=-4$, $b=(2-\sqrt{3})(2+\sqrt{3})=1$. Thus, $b-a=5$.

25. Ans: 2011

$$\begin{aligned} (x-2004)(x-2006)=2^y &\Longrightarrow (x-2005+1)(x-2005-1)=2^y \\ &\Longrightarrow (x-2005)^2-1=2^y \\ &\Longrightarrow (x-2005)^2=1+2^y. \end{aligned}$$

Write $n=|x-2005|$. Then we have

$$n^2=1+2^y\Longrightarrow 2^y=(n-1)(n+1).$$

Observe that the above equality implies easily that $n \neq 0, 1$. Thus, $n-1$ and $n+1$ are positive integers, and both $n-1$ and $n+1$ are powers of 2. Write $n-1 = 2^a$. Then

$$n + 1 = 2^a + 2 = 2(2^{a-1} + 1).$$

Since $n \geq 2$, it follows that $n+1 \geq 3$. Moreover, since $n+1$ is a power of 2, it follows that $n+1 \geq 4$ and thus $\frac{n+1}{2} \geq 2$. Hence $2^{a-1}+1$ is a positive integer and it is also a power of 2. Clearly, $2^{a-1}+1 > 1$. Thus $2^{a-1}+1 \geq 2 \Longrightarrow a \geq 1$. If $a > 1$, then $2^{a-1}+1 > 2$ and it is odd, which is not possible. Thus we have $a = 1$. Thus,

$$n = 1 + 2 = 3.$$

Then

$$2^y = (3-1)(3+1) = 8 \Longrightarrow y = 3.$$

It follows that

$$\begin{aligned}(x - 2005)^2 = 1 + 2^3 = 9 = 3^2 &\Longrightarrow x = 2005 \pm 3 \\ &\Longrightarrow x = 2008 \text{ or } 2002.\end{aligned}$$

Thus the largest possible value of $x + y$ is $2008 + 3 = 2011$ when $x = 2008$.

26. Ans: 30

Produce BC to F such that $CF = DE$. Then $\triangle BDE$ and $\triangle ACF$ are right-angled triangles. Note that $BE = AC$ and $DE = CF$. Thus, we have

$$\triangle BDE \cong \triangle ACF.$$

Therefore, $AF = BD = \frac{1}{2}$, and $\angle FAC = \angle ABC$. Since $\angle BAC + \angle ABC = 90°$, we have

$$\angle BAC + \angle FAC = 90°.$$

Hence, $\angle BAF = 90°$. Now in $\triangle BAF$, $AF = \frac{1}{2}$, and

$$BF = BC + CF = BC + DE = 1.$$

Hence,

$$\sin x° = \frac{AF}{BF} = \frac{1}{2} \Longrightarrow x = 30.$$

Alternative solution: Let $BE = a$ and $DE = b$. Since $\angle BED = 90°$, it follows from Pythagoras Theorem that

$$a^2 + b^2 = \frac{1}{4}.$$

Since $DE + BC = 1$, we have $BC = 1 - b$. As $\triangle BDE \sim \triangle ACF$, we have

$$\frac{b}{a} = \frac{a}{1-b} \Longrightarrow a^2 = b(1-b).$$

Together with the previous equality, we have

$$b(1-b) + b^2 = \frac{1}{4} \Longrightarrow b = \frac{1}{4}$$

Hence,

$$\sin x^\circ = \frac{DE}{BD} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \Longrightarrow x = 30.$$

27. Ans: 5

By the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, we have

$$\frac{1}{a^2 + ab + b^2} = \frac{a-b}{a^3 - b^3} \text{ if } a \neq b.$$

Let $a = \sqrt[3]{x+1}$ and $b = \sqrt[3]{x-1}$. Then

$$\begin{aligned} f(x) &= \frac{1}{(\sqrt[3]{x+1})^2 + (\sqrt[3]{x+1})(\sqrt[3]{x-1}) + (\sqrt[3]{x-1})^2} \\ &= \frac{\sqrt[3]{x+1} - \sqrt[3]{x-1}}{(x+1) - (x-1)} \\ &= \frac{1}{2}(\sqrt[3]{x+1} - \sqrt[3]{x-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} & f(1) + f(3) + \cdots + f(997) + f(999) \\ =& \frac{1}{2}(\sqrt[3]{2} - 0 + \sqrt[3]{4} - \sqrt[3]{2} + \cdots + \sqrt[3]{998} - \sqrt[3]{996} + \sqrt[3]{1000} - \sqrt[3]{998}) \\ =& \frac{1}{2} \times \sqrt[3]{1000} = \frac{10}{2} = 5. \end{aligned}$$

28. Ans: 20

Suppose that the area of $\triangle PQT$ is t cm^2. Since $PT = TS$, we see that

$$\text{area of } \triangle QTS = \text{area of } \triangle PQT = t \text{ cm}^2. \tag{1}$$

Suppose the area of $\triangle PTU$ is y cm^2. Then a similar argument shows that

$$\text{area of } \triangle STU = \text{area of } \triangle PTU = y \text{ cm}^2. \tag{2}$$

Given also that $QS = 2RS$, we have

$$\text{area of } \triangle SRU = \frac{1}{2} \times \text{ area of } \triangle QSU = \frac{1}{2}(t+y). \qquad (3)$$

Likewise, we have

$$\text{area of } \triangle PQS = 2 \times \text{ area of } \triangle PRS,$$

that is,

$$2t = 2\left(2y + \frac{t+y}{2}\right)$$

or

$$t = 5y. \qquad (4)$$

Also,

$$\begin{aligned} \text{the total area of } \triangle PQR &= \left(2t + 2y + \frac{t+y}{2}\right) \text{ cm}^2 \\ &= 150 \text{ cm}^2. \end{aligned}$$

Thus we have

$$t + y = 60. \qquad (5)$$

From (4) and (5), we obtain $y = 10$. So the area of $\triangle PSU$ is 20 cm^2.

29. Ans: 3

Let p and q be the roots of $x^2 - x - 1 = 0$. Then $p + q = 1$ and $pq = -1$. On the other hand, p and q are also roots of $ax^5 + bx^4 + 1 = 0$. Thus $ap^5 + bp^4 + 1 = 0$ and $aq^5 + bq^4 + 1 = 0$. From these two equations and $pq = -1$, we have $ap + b = -q^4$ and $aq + b = -p^4$. Therefore

$$\begin{aligned} a = \frac{p^4 - q^4}{p - q} &= (p^2 + q^2)(p + q) \\ &= ((p+q)^2 - 2pq)(p+q) \\ &= (1^2 - 2(-1)) \times 1 \\ &= 3. \end{aligned}$$

Alternative solution: Since $x^2 - x - 1$ is a factor of $ax^5 + bx + 1$, we may write

$$ax^5 + bx^4 + 1 \equiv (x^2 - x - 1)(ax^3 + cx^2 + dx - 1)$$

for some real numbers c and d. By comparing the coefficients, we have

$$\begin{aligned} x: &\quad 0 = 1 - d \Longrightarrow d = 1. \\ x^2: &\quad 0 = -1 - d - c \Longrightarrow c = -2 \\ x^3: &\quad 0 = -a - c + d \Longrightarrow a = -c + d = 2 + 1 = 3. \end{aligned}$$

30. Ans: 12

$$\begin{aligned}\sin^3\theta - \cos^3\theta &= (\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta)\\ &= \frac{\sqrt{6}-\sqrt{2}}{2}(1+\sin\theta\cos\theta).\end{aligned}$$

Now we have

$$\begin{aligned}(\sin\theta-\cos\theta)^2 = \left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2 &\Longrightarrow 1-2\sin\theta\cos\theta = \frac{6+2-2\sqrt{12}}{4} = 2-\sqrt{3}\\ &\Longrightarrow \sin\theta\cos\theta = \frac{\sqrt{3}-1}{2}.\end{aligned}$$

Hence we have

$$\begin{aligned}\sin^3\theta-\cos^3\theta &= \frac{\sqrt{6}-\sqrt{2}}{2}\left(1+\frac{\sqrt{3}-1}{2}\right)\\ &= \frac{\sqrt{2}(\sqrt{3}-1)}{2}\left(\frac{\sqrt{3}+1}{2}\right)\\ &= \frac{\sqrt{2}(3-1)}{4} = \frac{\sqrt{2}}{2}.\end{aligned}$$

Therefore,

$$24(\sin^3\theta-\cos^3\theta)^2 = 24\left(\frac{\sqrt{2}}{2}\right)^2 = 24\times\frac{1}{2} = 12.$$

31. Ans: 8

$$\begin{aligned}&x\sqrt{y}+y\sqrt{x}+\sqrt{2006xy}-\sqrt{2006x}-\sqrt{2006y}-2006=0\\ \Longleftrightarrow&(\sqrt{x}+\sqrt{y}+\sqrt{2006})(\sqrt{xy}-\sqrt{2006})=0\\ \Longleftrightarrow&\sqrt{xy}-\sqrt{2006}=0 \quad (\text{since } \sqrt{x}+\sqrt{y}+\sqrt{2006}>0)\\ \Longleftrightarrow&xy=2006=17\times59\times2.\end{aligned}$$

Thus the solutions are

$$(1,2006),(2006,1),(2,1003),(1003,2),(34,59),(59,34),(17,118),(118,17).$$

32. Ans: 375

Let $N = 1\times3\times\cdots\times2005$. We need to find the remainder when N is divided by 1000. Observe that $1000 = 8\times125$. Let $M = \frac{N}{125}$. We are going to find the remainder when M is divided by 8. Observe that

$$\begin{aligned}(2n-3)(2n-1)(2n+1)(2n+3) &= (4n^2-1)(4n^2-9)\\ &= 16n^4-40n^2+9\\ &\equiv 1 \pmod 8.\end{aligned}$$

Thus the product P of any 4 consecutive odd integers satisfies $P \equiv 1 \pmod 8$. Write

$$M = (1 \times 3 \times \cdots \times 123) \times (127 \times 129 \times \cdots \times 2005).$$

There are 62 factors in the expression $1 \times 3 \times \cdots \times 123$. Thus,

$$\begin{aligned} 1 \times 3 \times 5 \times \cdots \times 123 &= 1 \times 3 \times (5 \times \cdots \times 123) \\ &\equiv 3 \times 1^{15} \pmod 8 \\ &\equiv 3 \pmod 8. \end{aligned}$$

Similarly, there are 940 factors in the expression $127 \times \cdots \times 2005$. Thus,

$$\begin{aligned} 127 \times 129 \times \cdots \times 2005 &\equiv 1^{235} \pmod 8 \\ &\equiv 1 \pmod 8. \end{aligned}$$

Hence,

$$M \equiv 1 \times 3 \equiv 3 \pmod 8.$$

In other words,

$$M = 8n + 3$$

for some positive integer n. Now, we have

$$\begin{aligned} N &= 125 \times M \\ &= 125 \times (8n + 3) \\ &= 1000n + 375. \end{aligned}$$

Therefore, the remainder is 375.

33. Ans: 4305

Letting $y = 1$, one gets

$$f(x+1) = \left(1 + \frac{1}{x+1}\right) f(x) + \left(1 + \frac{x}{2}\right)\frac{3}{2} + x^2 + 2x.$$

Upon rearranging, one gets

$$\frac{f(x+1)}{x+2} - \frac{f(x)}{x+1} = x + \frac{3}{4}.$$

Then we have

$$\begin{aligned} \frac{f(n)}{n+1} - \frac{f(n-1)}{n} &= n - 1 + \frac{3}{4}, \\ \frac{f(n-1)}{n} - \frac{f(n-2)}{n-1} &= n - 2 + \frac{3}{4}, \\ &\vdots \\ \frac{f(2)}{3} - \frac{f(1)}{2} &= 1 + \frac{3}{4}. \end{aligned}$$

Adding these equalities together, we get

$$\begin{aligned}\frac{f(n)}{n+1} - \frac{f(1)}{2} &= 1 + 2 + \cdots + (n-1) + \frac{3}{4}(n-1) \\ &= \frac{(n-1)n}{2} + \frac{3}{4}(n-1).\end{aligned}$$

Thus,

$$f(n) = (n+1)\left[\frac{(n-1)n}{2} + \frac{3}{4}(n-1) + \frac{1}{2}\cdot\frac{3}{2}\right] = \frac{n(n+1)(2n+1)}{4}.$$

Hence,

$$f(20) = \frac{(20)(21)(41)}{4} = 4305.$$

34. Ans: 250

Observe that when $n \geq 2$,

$$\begin{cases} nx_n &= -1000(x_0 + x_1 + \cdots + x_{n-1}), \\ (n-1)x_{n-1} &= -1000(x_0 + x_1 + \cdots + x_{n-2}). \end{cases}$$

Thus, we have

$$nx_n - (n-1)x_{n-1} = -1000x_{n-1} \Longrightarrow x_n = -\left(\frac{1000-(n-1)}{n}\right)x_{n-1}.$$

It is easy to check that the above formula holds even when $n = 1$. Therefore, for $1 \leq n \leq 100$, we have

$$\begin{aligned} x_n &= -\frac{1000-(n-1)}{n}x_{n-1} \\ &= (-1)^2\frac{1000-(n-1)}{n}\cdot\frac{1000-(n-2)}{n-1}x_{n-2} \\ &= \cdots \\ &= (-1)^n\frac{1000-(n-1)}{n}\cdot\frac{1000-(n-2)}{n-1}\cdots\frac{1000}{1}x_0 \\ &= (-1)^n\binom{1000}{n}x_0. \end{aligned}$$

Hence we have

$$\begin{aligned}
&\frac{1}{2^2}x_0 + \frac{1}{2}x_1 + x_2 + \cdots + 2^{998}x_{1000}\\
&= \frac{1}{4}(x_0 + 2x_1 + 2^2x_2 + \cdots + 2^{1000}x_{1000})\\
&= \frac{1}{4}\left(x_0 - \binom{1000}{1}2x_0 + \binom{1000}{2}2^2x_0 - \binom{1000}{3}2^3x_0 + \cdots + \binom{1000}{1000}2^{1000}x_0\right)\\
&= \frac{x_0}{4}\left(1 - \binom{1000}{1}2 + \binom{1000}{2}2^2 - \binom{1000}{2}2^3 + \cdots + \binom{1000}{1000}2^{1000}\right)\\
&= \frac{1000}{4}(1-2)^{1000}\\
&= 250.
\end{aligned}$$

35. Ans: 76

Let u, v be the two positive integral solutions of the given equation. Then

$$\begin{cases} u + v = p, \\ uv = \dfrac{66p - 1}{5}. \end{cases}$$

Upon eliminating p, we have

$$\begin{aligned}
5uv = 66(u+v) - 1 &\Longrightarrow v(5u - 66) = 66u - 1 > 0\\
&\Longrightarrow 5u - 66 > 0, \text{ since } v > 0.
\end{aligned}$$

Similarly, we have $5v - 66 > 0$. Moreover, we have

$$\begin{aligned}
(5u - 66)(5v - 66) &= 25uv - 330(u+v) + 66^2\\
&= 25 \times \left(\frac{66p-1}{5}\right) - 330p + 66^2\\
&= -5 + 66^2 = 4351 = 19 \times 229.
\end{aligned}$$

Without loss of generality, we may assume that $u \geq v$. Since both 19 and 229 are prime, we must have

$$\begin{cases} 5u - 66 &= 229 \\ 5v - 66 &= 19 \end{cases} \quad \text{or} \quad \begin{cases} 5u - 66 &= 4351 \\ 5v - 16 &= 1. \end{cases}$$

The first set of equations imply $u = 59$, $v = 17$. The second set of equations does not have integral solutions. Hence, we must have $u = 59$ and $v = 17$. Thus, $p = u + v = 76$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Senior Section, Special Round)

Saturday, 24 June 2006 0930– 1230

Important:

Attempt as many questions as you can.

No calculators are allowed.

Show all the steps in your working.

Each question carries 10 marks.

1. Let a, d be integers such that $a, a+d, a+2d$ are all prime numbers larger than 3. Prove that d is a multiple of 6.

2. Let $ABCD$ be a cyclic quadrilateral, let the angle bisectors at A and B meet at E, and let the line through E parallel to side CD intersect AD at L and BC at M. Prove that $LA + MB = LM$.

3. Two circles are tangent to each other internally at a point T. Let the chord AB of the larger circle be tangent to the smaller circle at a point P. Prove that the line TP bisects $\angle ATB$.

4. You have a large number of congruent equilateral triangular tiles on a table and you want to fit n of them together to make a convex equiangular hexagon (i.e., one whose interior angles are $120°$). Obviously, n cannot be any positive integer. The first three feasible n are 6, 10 and 13. Determine if 19 and 20 are feasible.

5. It is claimed that the number

$$N = 526315789473684210$$

is a *persistent* number, that is, if multiplied by any positive integer the resulting number always contains the ten digits $0, 1, \ldots, 9$ in some order with possible repetitions.

(a) Prove or disprove the claim.

(b) Are there any persistent numbers smaller than the above number?

Singapore Mathematical Society

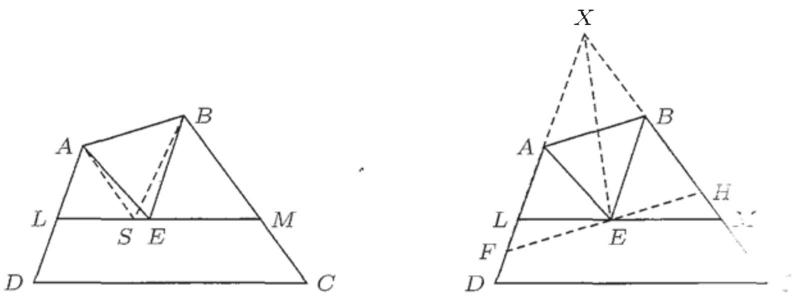
Singapore Mathematical Olympiad (SMO) 2006

(Senior Section, Special Round Solutions)

1. First note that d is even. Primes larger than 3 are of the form $6n+1$ or $6n+5$. Thus two of the three primes are of the same form. Their difference is either d or $2d$ and is divisible by 6. Thus d is divisible by 3 is hence divisible by 6.

2. *First solution*: Assume that $x > y$ (See Fig.(i)). Choose a point S on the segment LM so that $LS = LA$. Clearly $\angle ASL = \angle LAS = y$. Therefore, $ASEB$ is cyclic. As $\angle LAS = y < x = \angle LAE$, it follows that S is between L and E.

On the other hand, $\angle SBM = \angle SBE + \angle EBM = \angle SAE + \angle EBM = \angle LAE - \angle LAS + y = x - y + y = x = \angle BAM$. Hence, MBS is isosceles and $MS = MB$. Therefore, $LM = LS + SM = LA + MB$.



(i) (ii)

Second solution: Produce DA and CB to meet at X (See Fig.(ii)). Draw FH parallel to AB. Draw XE with F on AD and H on BC. Then E is an excentre of $\triangle XAB$ and so XE bisects $\angle AXB$. $\triangle MLX \equiv \triangle HFX$ since they have equal angles and a common angle bisector. Hence $ME = FE$, $HE = LE$ and $HM = LF$. Since FH is parallel to AB, and AE bisects $\angle DAB$, $\angle FAE = \angle AEF$. Thus $\triangle FAE$ is isosceles, and $AF = FE$. Similarly, $BH = EH$. The rest follows easily.

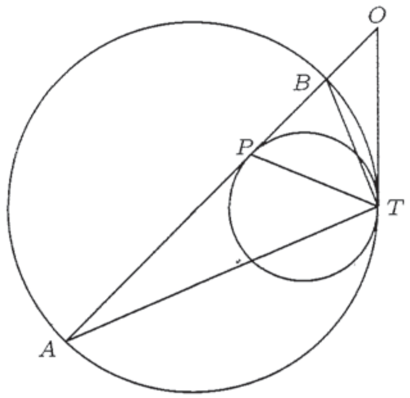
Third solution: Note that

$$\sin 2x + \sin 2y = 2\sin(2y - x)\cos x + 2\sin(2x - y)\cos y.$$

In the quadrilateral $ABML$, let $2x = \angle A$, $2y = \angle B$. Therefore $\angle L = 180° - 2y$ and $\angle M = 180° - 2x$. Thus $\angle LEA = 2y - x$, $\angle BEM = 2x - y$. If d is the common distance from E to AL, AB and BM, then the result follows from

$$\begin{aligned}\frac{LE+EM-LA-MB}{d} &= \frac{1}{\sin 2y}+\frac{1}{\sin 2x}-\frac{\sin(2y-x)}{\sin x\sin 2y}-\frac{\sin(2x-y)}{\sin y\sin 2x}\\ &= \frac{\sin 2x+\sin 2y}{\sin 2x\sin 2y}-2\frac{\sin(2y-x)\cos x+\sin(2x-y)\cos y}{\sin 2x\sin 2y}\\ &= 0\end{aligned}$$

3. Let the tangent at T meet the extension of the chord AB at O. Then $\angle BTO = \angle TAB$. Thus $\triangle OAT$ is similar to $\triangle OTB$ so that $\frac{TA}{TB}=\frac{OT}{OB}$. Since $OT = OP$, we have $\frac{TA}{TB}=\frac{OP}{OB}$. On the other hand, $OP^2 = OA \cdot OB$.



Therefore $\frac{TA}{TB}=\frac{OP}{OB}=\frac{OA}{OP}$. Thus $\frac{TA}{TB}=\frac{OA-OP}{OP-OB}=\frac{AP}{BP}$. Using the angle bisector theorem, we see that TP bisects $\angle ATB$.

4. See Junior Section Special Round Problem 5. You just need to check further the cases for $\ell = 5, 6$.

5. The fact is there are no persistent numbers. For any positive integer N, consider the remainder when following N numbers are divisible by N:

$$1, 11, 111, \ldots, \underbrace{11\ldots1}_{N}$$

If one of the remainders is 0, N is not persistent. If not, then two of the remainders are the same. Thus there exist two, say A, B such that $A - B = 11\ldots100\ldots0$ is divisible by N, again N is not persistent.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Open Section, Round 1)

Wednesday, 31 May 2006 | 0930-1200

Important:

Answer ALL 25 questions.

Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

1. How many integers are there between 0 and 10^5 having the digit sum equal to 8?

2. Given that p and q are integers that satisfy the equation $36x^2 - 4(p^2+11)x + 135(p+q) + 576 = 0$, find the value of $p+q$.

3. A function f is such that $f : \mathbb{R} \to \mathbb{R}$ where $f(xy+1) = f(x)f(y) - f(y) - x + 2$ for all $x, y \in \mathbb{R}$. Find $10f(2006) + f(0)$.

4. Three people A, B and C play a game of passing a basketball from one to another. Find the number of ways of passing the ball starting with A and reaching A again on the 11th pass. For example, one possible sequence of passing is

$$A \to B \to A \to B \to C \to A \to B \to C \to B \to C \to B \to A.$$

5. There are 10! permutations $s_0 s_1 \ldots s_9$ of $0, 1, \ldots, 9$. How many of them satisfy $s_k \geq k-2$ for $k = 0, 1, \ldots, 9$?

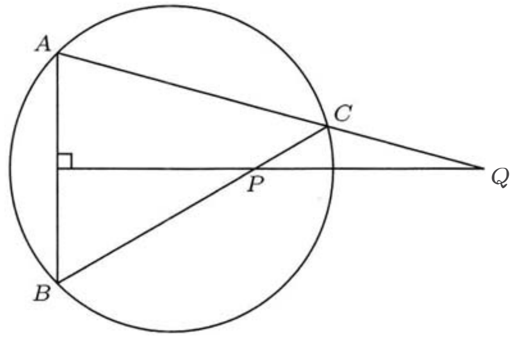
6. A triangle $\triangle ABC$ has its vertices lying on a circle $\mathbb{C}$ of radius 1, with $\angle BAC = 60°$. A circle with center I is inscribed in $\triangle ABC$. The line AI meets circle $\mathbb{C}$ again at D. Find the length of the segment ID.

7. Find the number of consecutive 0's at the end of the base 10 representation of 2006!.

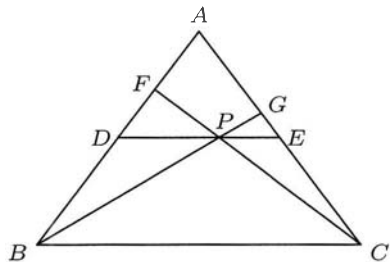
8. For any non-empty finite set A of real numbers, let $s(A)$ be the sum of the elements in A. There are exactly 61 3-element subsets A of $\{1, \ldots, 23\}$ with $s(A) = 36$. Find the number of 3-element subsets of $\{1, \ldots, 23\}$ with $s(A) < 36$.

9. Suppose f is a function satisfying $f(x + x^{-1}) = x^6 + x^{-6}$, for all $x \neq 0$. Determine $f(3)$.

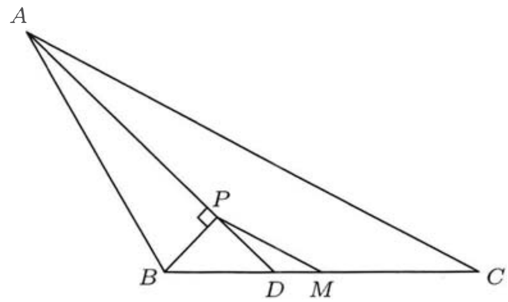
10. Points A, B, C lie on a circle centered at O with radius 7. The perpendicular bisector of AB meets the segment BC at P and the extension of AC at Q. Determine the value of $OP \cdot OQ$.



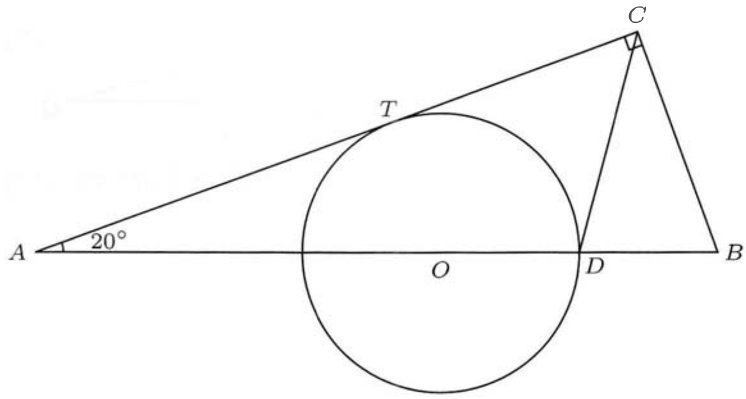
11. In the triangle ABC, $AB = AC = 1$, D and E are the midpoints of AB and AC respectively. Let P be a point on DE and let the extensions of BP and CP meet the sides AC and AB at G and F respectively. Find the value of $\frac{1}{BF} + \frac{1}{CG}$.



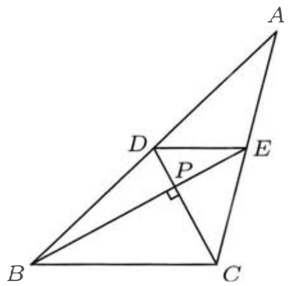
12. In the triangle ABC, $AB = 14, BC = 16, AC = 26$, M is the midpoint of BC and D is the point on BC such that AD bisects $\angle BAC$. Let P be the foot of the perpendicular from B onto AD. Determine the length of PM.



13. In the triangle ABC, $\angle A = 20°$, $\angle C = 90°$, O is a point on AB and D is the midpoint of OB. Suppose the circle centered at O with radius OD touches the side AC at T. Determine the size of $\angle BCD$ in degrees.



14. In $\triangle ABC$, D and E are the midpoints of the sides AB and AC respectively, CD and BE intersect at P with $\angle BPC = 90°$. Suppose $BD = 1829$ and $CE = 1298$. Find BC.



15. Let $X = \{1, 2, 3, \ldots, 17\}$. Find the number of subsets Y of X with odd cardinalities.

16. Find the value of $400(\cos^5 15° + \sin^5 15°) \div (\cos 15° + \sin 15°)$.

17. Find the number of real solutions of the equation

$$x^2 + \frac{1}{x^2} = 2006 + \frac{1}{2006}.$$

18. Find the largest integer n such that n is a divisor of $a^5 - a$ for all integers a.

19. Given two sets $A = \{1, 2, 3, \ldots, 15\}$ and $B = \{0, 1\}$, find the number of mappings $f : A \to B$ with 1 being the image of at least two elements of A.

20. Let $a_1, a_2, \ldots$ be a sequence satisfying the condition that $a_1 = 1$ and $a_n = 10a_{n-1} - 1$ for all $n \geq 2$. Find the minimum n such that $a_n > 10^{100}$.

21. Let P be a 30-sided polygon inscribed in a circle. Find the number of triangles whose vertices are the vertices of P such that any two vertices of each triangle are separated by at least three other vertices of P.

22. A year is called a leap year if it is either divisible by 4 but not divisible by 100, or divisible by 400. Hence, the years 2000, 2004 and 2400 are leap years while the years 2006, 2100 and 2200 are not. Find the number of leap years between 2000 and 4000 inclusive.

23. The birth date of Albert Einstein is 14 March 1879. If we denote Monday by 1, Tuesday by 2, Wednesday by 3, Thursday by 4, Friday by 5, Saturday by 6 and Sunday by 7, which day of the week was Albert Einstein born? Give your answer as an integer from 1 to 7.

24. Find the number of 7-digit integers formed by some or all of the five digits, namely, 0, 1, 2, 3, and 4, such that these integers contain none of the three blocks 22, 33 and 44.

25. Let

$$S = \sum_{r=0}^{n} \binom{3n + r}{r}.$$

Evaluate $S \div (23 \times 38 \times 41 \times 43 \times 47)$ when $n = 12$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Open Section, Round 1 Solutions)

1. Ans: 495

Each integer can be written as $\overline{x_1x_2x_3x_4x_5}$ where each $x_t = 0, 1, 2, \ldots, 9$ with $x_1 + x_2 + x_3 + x_4 + x_5 = 8$. The number of non-negative integer solutions to the above equation is 495. So there are 495 such integers.

2. Ans: 20

$$p + q = \frac{p^2 + 11}{9} \text{ and } pq = \frac{135(p+q) + 576}{36} = \frac{15(p+q)}{4} + 16.$$

So $p + q > 0$ and is a multiple of 4. Also $p + q = 1 + \frac{p^2+2}{9}$. So $p^2 + 2$ is a multiple of 9. So $p = 5, 13, 14, \ldots$. If $p = 5$, $p + q = 4$ and $pq = 31$. If $p = 13$, $p + q = 20$, $pq = 91$, $q = 7$. Thus $p + q = 20$.

3. Ans: 20071

$f(xy + 1) = f(x)f(y) - f(y) - x + 2$, so we have $f(yx + 1) = f(y)f(x) - f(x) - y + 2$. Subtracting, we have $0 = f(x) - f(y) + y - x$ or $f(x) + y = f(y) + x$. Let $y = 0$. Then $f(x) = f(0) + x$. Substitute into the given identity and putting $x = y = 0$, we get

$$f(0) + 1 = f(0)f(0) - f(0) + 2, \quad \text{or} \quad (f(0) - 1)^2 = 0.$$

Thus $f(0) = 1$ and $10f(2006) + f(0) = 20071$.

4. Ans: 682

Let a_k denote the number of ways that the k^{th} pass reach A. We have $a_1 = 0$. At each pass, the person holding the ball has 2 ways to pass the ball to. So total number of ways the ball can be passed after the k^{th} pass is 2^k The number of ways that at the $(k+1)^{\text{th}}$ pass, A receives the ball is $a_k + 1$. So $a_{k+1} = 2^k - a_k$ Thus $a_1 = 0$, $a_2 = 2$, $a_3 = 2, \ldots, a_{11} = 682$.

5. Ans: 13122

Construct the permutation from the end. There are 3 choices each for s_9, s_8, $\ldots$, s_2 and 2 choices for s_1 and 1 choice for s_0. So the answer is $2.3^8 = 13122$.

6. Ans: 1

AD bisects the angle $\angle A$ and IC bisects the angle $\angle C$. Now $\angle BCD = \angle BAD = \angle A/2$. $\angle ADC = \angle B$. Hence $\angle ICD = (\angle A + \angle C)/2$ and $\angle DIC = 180° - \angle B - (\angle A + \angle C)/2 = (\angle A + \angle C)/2$. Thus $ID = CD$. The chord CD subtends an angle of $30°$ at point A of circle $\mathbb{C}$. Hence it subtends an angle of $60°$ at the center of circle $\mathbb{C}$. Thus $ID = CD = 2\sin(60°/2) = 1$.

7. Ans: 500

If p is a prime, then the highest power of p that divides 2006! is

$$f(p) = \lfloor 2006/p \rfloor + \lfloor 2006/p^2 \rfloor + \lfloor 2006/p^3 \rfloor + \cdots.$$

(Note that the terms in the sum are eventually 0.) The number of consecutive 0's at the end of the base 10 representation of 2006! is the highest power of 10 that divides 2006!, which is $\min\{f(2), f(5)\} = f(5) = 401 + 80 + 16 + 3 = 500$.

8. Ans: 855

The map $\{a, b, c\} \to \{24-a, 24-b, 24-c\}$ is a bijection from the set of 3-element subsets of $\{1, \ldots, 23\}$ with $s(A) < 36$ onto the set of 3-element subsets of $\{1, \ldots, 23\}$ with $s(A) > 36$. The number of 3-element subsets of $\{1, \ldots, 23\}$ is $\binom{23}{3} = 1771$. Therefore, the number of 3-element subsets of $\{1, \ldots, 23\}$ with $s(A) < 36$ is

$$\frac{1}{2}\left[1771 - \text{number of 3-element subsets of } \{1, \ldots, 23\} \text{ with } s(A) = 36\right] = 855.$$

9. Ans: 322

Note that $x^6 + x^{-6} = (x^2 + x^{-2})(x^4 - 1 + x^{-4}) = ((x + x^{-1})^2 - 2)((x^2 + x^{-2})^2 - 3) = ((x + x^{-1})^2 - 2)(((x + x^{-1})^2 - 2)^2 - 3) = f(x + x^{-1})$. Thus letting $z = x + x^{-1}$, we have $f(z) = (z^2 - 2)((z^2 - 2)^2 - 3)$. Therefore, $f(3) = (3^2 - 2)((3^2 - 2)^2 - 3) = 322$.

10. Ans: 49

Let R be the foot of the perpendicular from O to BC. Since $\angle BAC = \angle COR$, we have $\angle AQO = \angle OCP$. Thus $\triangle COP$ is similar to $\triangle QOC$. Therefore $OC/OQ = OP/OC$ so that $OP \cdot OQ = OC^2 = 7^2 = 49$.

11. Ans: 3

Let $DP = x$, $PE = y$, and $BC = a$. As $\triangle FDP$ is similar to $\triangle FBC$, we have $DF/BF = x/a$. Thus $(BF - 1/2)/BF = x/a$. Solving for BF, we have $1/BF = 2(a - x)/a$.

Similarly, from the fact that $\triangle GPE$ is similar to $\triangle GBC$, we obtain $1/CG = 2(a - y)/a$. Consequently,

$$\frac{1}{BF} + \frac{1}{CG} = (4a - 2(x + y))/a = (4a - a)/a = 3.$$

12. Ans: 6

Extend BP meeting AC at E. Then ABE is an isosceles triangle with $AB = AE$ and $BP = PE$. As P and M are the midpoints of BE and BC respectively, we have PM is parallel to EC and $PM = EC/2 = (26 - 14)/2 = 6$.

13. Ans: 35

Join DT and DC. Let M be the foot of the perpendicular from D onto AC. Then OT, DM and BC are all parallel. Since D is the midpoint of OB, M is the midpoint of TC. Thus DTC is an isosceles triangle with $DT = DC$ and $\angle TDM = \angle MDC$.

As $\angle TOA = 70°$ and $\triangle OTD$ is isosceles, we have $\angle OTD = 35°$. Thus $\angle BCD = \angle MDC = \angle TDM = \angle OTD = 35°$.

14. Ans: 2006

Since D and E are the midpoints of the sides AB and AC respectively, we have DE is parallel to BC and $\triangle PDE$ is similar to $\triangle PCB$ with $PD : PC = PE : PB = DE : CB = 1 : 2$.

Let $PE = x$ and $PD = y$. Then $PB = 2x$ and $PC = 2y$. By Pythagoras' theorem applied to $\triangle PBD$ and $\triangle PCE$, we get $(2x)^2 + y^2 = 1829^2$ and $(2y)^2 + x^2 = 1298^2$. Adding these two equations, we have $x^2 + y^2 = (1829^2 + 1298^2)/5$. Thus $BC^2 = (2x)^2 + (2y)^2 = 4(1829^2 + 1298^2)/5 = 4024036$. Therefore $BC = \sqrt{4024036} = 2006$.

15. Ans: 65536

The answer is

$$\binom{17}{1} + \binom{17}{3} + \binom{17}{5} + \cdots + \binom{17}{17} = 2^{16} = 65536.$$

16. Ans: 275

$$
\begin{aligned}
&400(\cos^5 15^\circ + \sin^5 15^\circ) \div (\cos 15^\circ + \sin 15^\circ)\\
&= 400(\cos^4 15^\circ - \cos^3 15^\circ \sin 15^\circ + \cos^2 15^\circ \sin^2 15^\circ - \cos 15^\circ \sin^3 15^\circ + \sin^4 15^\circ)\\
&= 400(\cos^4 15^\circ + \sin^4 15^\circ - \cos 15^\circ \sin 15^\circ + \cos^2 15^\circ \sin^2 15^\circ)\\
&= 400((\cos^2 15^\circ + \sin^2 15^\circ)^2 - \cos^2 15^\circ \sin^2 15^\circ - \cos 15^\circ \sin 15^\circ)\\
&= 400(1 - (\frac{1}{2}\sin 30^\circ)^2 - (\frac{1}{2}\sin 30^\circ))\\
&= 400(1 - 1/16 - 1/4)\\
&= 275
\end{aligned}
$$

17. Ans: 4

Consider the equation $y + 1/y = a$, where $a > 0$. It can be changed into

$$y^2 - ay + 1 = 0.$$

Observe that it has two positive real solutions:

$$y = \frac{a \pm \sqrt{a^2 - 4}}{2} > 0.$$

Thus the equation

$$x^2 + \frac{1}{x^2} = 2006 + \frac{1}{2006}.$$

has four real solutions (i.e., $\pm\sqrt{2006}, \pm 1/\sqrt{2006}$).

18. Ans: 30

Note that

$$a^5 - a = a(a-1)(a+1)(a^2+1).$$

It is clear that $2 \mid a^5 - a$ and $3 \mid a^5 - a$. We can show that $5 \mid a^5 - a$ by considering the five cases: $a \equiv i \pmod 5$, $i = 0, 1, 2, 3, 4$. Thus $30 \mid a^5 - a$.

When $a = 2$, we have $a^5 - a = 30$. Thus the maximum n is 30.

19. Ans: 32752

There are 2^{15} mappings from A to B. There is only one mapping $f : A \to B$ with $f(i) = 0$ for all $i \in A$; and there are 15 mappings $f : A \to B$ with $f(i) = 0$ for all $i \in A \backslash \{k\}$ and $f(k) = 1$, for $k = 1, 2, \ldots, 15$. Thus the answer is 32752.

20. Ans: 102

Note that from $a_n = 10a_{n-1} - 1$, we have

$$a_n - \frac{1}{9} = 10\left(a_{n-1} - \frac{1}{9}\right)$$

for all $n \geq 2$. Thus,

$$a_n - \frac{1}{9} = 10^{n-1}\left(a_1 - \frac{1}{9}\right) = 10^{n-1}\frac{8}{9}$$

for all $n \geq 1$. Therefore

$$a_n = \frac{(1 + 8 \times 10^{n-1})}{9}.$$

Observe that for $n \geq 2$,

$$8 \times 10^{n-2} < a_n < 10^{n-1}.$$

Thus

$$a_{101} < 10^{100} < a_{102}.$$

That is why the answer is 102.

21. Ans: 1900

Let A be a vertex of P. First we shall count the number of such triangles having A as a vertex. After taking away A and 3 consecutive vertices of P on each side of A, we are left with 23 vertices from which we can choose two vertices in such a way that, together with A, a desired triangle can be formed. There are $\binom{2+(23-3-2)}{2} = \binom{20}{2}$ ways to do so. Hence there are $30\binom{20}{2} \div 3 = 1900$ such triangles.

22. Ans: 486

Let $S = \{x \in \mathbb{Z} \mid 2000 \leq x \leq 4000\}$, $A = \{x \in S \mid x \text{ is divisible by } 4\}$, $B = \{x \in S \mid x \text{ is divisible by } 100\}$,$C = \{x \in S \mid x \text{ is divisible by } 400\}$. The required answer is

$$|A|-|B|+|C| = \left(\frac{4000}{4} - \frac{2000}{4} + 1\right) - \left(\frac{4000}{100} - \frac{2000}{100} + 1\right) + \left(\frac{4000}{400} - \frac{2000}{400} + 1\right) = 486.$$

23. Ans: 5

Our reference day is today, 31-5-2006, Wednesday. We shall first count the number of days D from 15-5-1879 to 31-5-2006. The number of leap years between 1879 and 2005 is

$$\left(\left\lfloor\frac{2005}{4}\right\rfloor - \left\lfloor\frac{1879}{4}\right\rfloor\right) - \left(\left\lfloor\frac{2005}{100}\right\rfloor - \left\lfloor\frac{1879}{100}\right\rfloor\right) + \left(\left\lfloor\frac{2005}{400}\right\rfloor - \left\lfloor\frac{1879}{400}\right\rfloor\right) = 31.$$

From 1-1-1880 to 31-12-2005 there are $2005 - 1879 = 126$ years, of which 31 are leap years. Thus $D = 95\times365+31\times366+(365-31-28-14)+(31+28+31+30+31) = 46464$. Since 46464 leaves a remainder of 5 when divided by 7, Albert Einstein was born on Friday. The answer is 5.

24. Ans: 29776

Let a_n denote the number of such n-digit integers. Among these a_n integers, let b_n denote the number of those which end with 2. By symmetry, the number of those which end with 3 (or 4) is also equal to b_n. Hence

$$a_n = \underbrace{2a_{n-1}}_{\text{end with 0 or 1}} + \underbrace{3b_n}_{\text{end with 2,3 or 4}} \tag{1}$$

$$b_n = \underbrace{2a_{n-2}}_{\text{end with 02 or 12}} + \underbrace{2b_{n-1}}_{\text{end with 32 or 42}} \tag{2}$$

Thus

$$\begin{aligned} a_n - 2a_{n-1} = 3b_n &= 6a_{n-2} + 6b_{n-1} \\ &= 6a_{n-2} + 2(a_{n-1} - 2a_{n-2}) \\ a_n &= 4a_{n-1} + 2a_{n-2} \end{aligned}$$

We have $a_1 = 4$, $a_2 = 4 \times 5 - 3 = 17$. By iterating we get $a_7 = 29776$.

25. Ans: 1274

By using the fact that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$, we have

$$\begin{aligned} S &= \binom{3n}{0} + \binom{3n+1}{1} + \cdots + \binom{3n+n}{n} = \binom{3n+1}{0} + \binom{3n+1}{1} + \cdots + \binom{3n+n}{n} \\ &= \cdots = \binom{3n+n}{n-1} + \binom{3n+n}{n} = \binom{4n+1}{n}. \end{aligned}$$

Thus when $n = 12$,

$$\frac{S}{23 \times 38 \times 41 \times 43 \times 47} = 1274.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Open Section, Special Round)

Saturday, 1 July 2006 0900– 1330

Important:

Attempt as many questions as you can.

No calculators are allowed.

Show all the steps in your working.

Each question carries 10 marks.

1. In the triangle ABC, $\angle A = 60°$, D, M are points on the line AC and E, N are points on the line AB such that DN and EM are the perpendicular bisectors of AC and AB respectively. Let L be the midpoint of MN. Prove that $\angle EDL = \angle ELD$.

2. Show that any representation of 1 as the sum of distinct reciprocals of numbers drawn from the arithmetic progression $\{2, 5, 8, 11, \ldots\}$ such as given in the following example must have at least eight terms:

$$1 = \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{20} + \frac{1}{41} + \frac{1}{110} + \frac{1}{1640}.$$

3. Consider the sequence $p_1, p_2, \ldots$ of primes such that for each $i \geq 2$, either $p_i = 2p_{i-1} - 1$ or $p_i = 2p_{i-1} + 1$. An example is the sequence $2, 5, 11, 23, 47$. Show that any such sequence has a finite number of terms.

4. Let n be a positive integer. Let $S_1, S_2, \cdots, S_k$ be a collection of $2n$-element subsets of $\{1, 2, 3, 4, \cdots, 4n-1, 4n\}$ so that $S_i \cap S_j$ contains at most n elements for all $1 \leq i < j \leq k$. Show that

$$k \leq 6^{(n+1)/2}.$$

5. Let a, b and n be positive integers. Prove that $n!$ divides

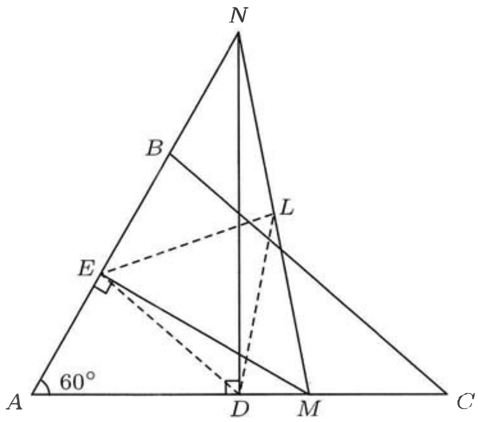
$$b^{n-1}a(a+b)(a+2b)\cdots(a+(n-1)b).$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Open Section, Special Round Solutions)

1. Set up a coordinate system so that A is the origin and AC is the x-axis. Let the coordinate of C be $(2c, 0)$ and the coordinate of B be $(2b, 2b\sqrt{3})$. Then the coordinates of D, E, N and M are $(c, 0)$, $(b, b\sqrt{3})$, $(c, c\sqrt{3})$ and $(4b, 0)$, respectively.



Thus
$$MN^2 = (c - 4b)^2 + (c\sqrt{3} - a)^2 = 4c^2 - 8bc + 16b^2.$$
Also
$$BC^2 = (2b - 2c)^2 + (2b\sqrt{3})^2 = 4c^2 - 8bc + 16b^2.$$
Therefore $MN = BC$. In the right-angled triangle EMN, $EL = \frac{1}{2}MN$. Thus $EL = \frac{1}{2}BC = ED$. That is $\angle EDL = \angle ELD$.

2. Suppose that the representation uses the reciprocals of k distinct positive integers, $x_1, \ldots, x_k$, where $x_i \equiv 2 \pmod 3$. Since $1 = \sum \frac{1}{x_i}$, we get
$$x_1 x_2 \ldots x_k = \sum X_i$$
where $X_i = \frac{x_1 x_2 \ldots x_n}{x_i}$. Thus
$$2^k \equiv k2^{k-1} \pmod 3,$$
from which we get $k \equiv 2 \pmod 3$. Hence $k = 2, 5, 8, \ldots$. Since
$$\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} < 1$$
we see that we need at least 8 terms.

3. Except for 2 or 3, each prime is of the form $6u \pm 1$. If a prime $p = 6u + 1$ is in the chain, its successor, if any, must be of the form $2p - 1 = 2(6u) + 1$ since $2p + 1$ is divisible by 3. Hence the successors are:

$$2(6u) + 1, 2^2(6u) + 1, \ldots, 2^i(6u) + 1, \ldots$$

This sequence cannot go on forever giving primes. To prove this claim, we first note that, by Fermat's Little Theorem, there exists k such that $2^k \equiv 1 \pmod{6u + 1}$. Thus

$$2^k(6u) + 1 \equiv 0 \pmod{6u + 1}.$$

Hence $2^k(6u) + 1$ is not prime. A similarly argument can be given for the case $6u + 1$.

4. Let $A = \{1, 2, \cdots, 4n\}$. Let $\mathcal{F}$ be the family of subsets in A with $n + 1$-elements. Then

$$|\mathcal{F}| = \binom{4n}{n+1}.$$

Note that every $n+1$-element in S_i is also a member in $\mathcal{F}$. Since $S_i \cap S_j$ contains at most n elements in A, and any $n+1$-element in S_i is different from any $n+1$-element in S_j for all $1 \leq i < j \leq k$. Thus

$$|\mathcal{F}| \geq \sum_{i=1}^{k} \binom{2n}{n+1} = k\binom{2n}{n+1}.$$

Hence

$$\begin{aligned} k &\leq \binom{4n}{n+1} \div \binom{2n}{n+1} \\ &= \frac{4n \times (4n-1) \times \cdots \times 3n}{2n \times (2n-1) \times \cdots \times n}. \end{aligned}$$

It can be shown that

$$\frac{(4n-i)(3n+i)}{(2n-i)(n+i)} \leq \frac{4n \times 3n}{2n \times n} = 6$$

for all $0 \leq i \leq (n-1)/2$.

If n is odd, then

$$\frac{4n \times (4n-1) \times \cdots \times 3n}{2n \times (2n-1) \times \cdots \times n} = \prod_{i=0}^{(n-1)/2} \frac{(4n-i)(3n+i)}{(2n-i)(n+i)} \leq 6^{(n+1)/2}.$$

If n is even, then

$$\frac{4n \times (4n-1) \times \cdots \times 3n}{2n \times (2n-1) \times \cdots \times n} = \frac{4n - n/2}{2n - n/2} \prod_{i=0}^{(n-2)/2} \frac{(4n-i)(3n+i)}{(2n-i)(n+i)} \leq 6^{1/2}6^{n/2} = 6^{(n+1)/2}.$$

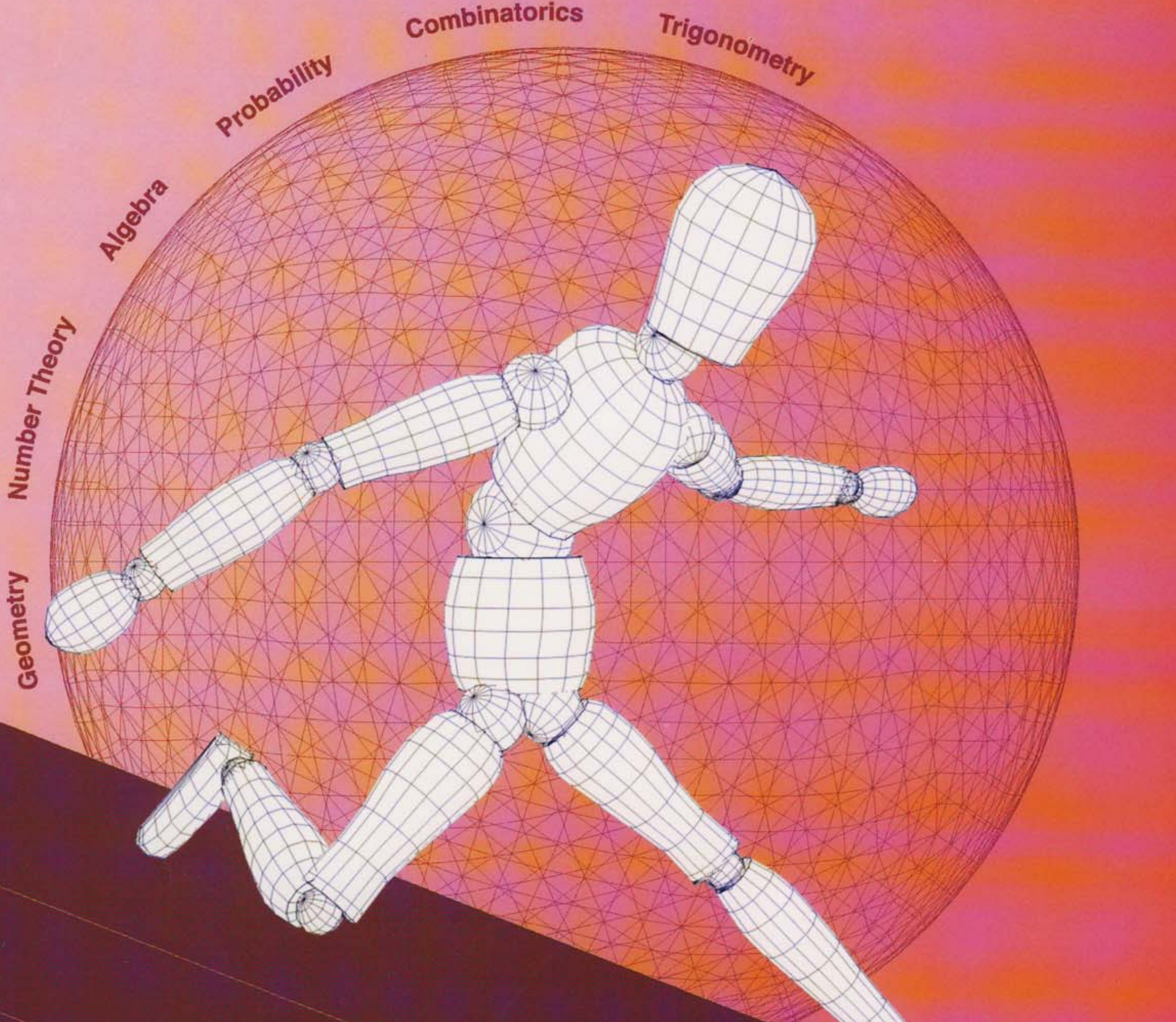
5. We shall prove that for any prime p with $1 < p \leq n$, if $p^\alpha \mid n!$, then $p^\alpha \mid b^{n-1}a(a+b)(a+2b)\cdots(a+(n-1)b)$.

   If $p \mid b$, then as
$$\alpha = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor < \sum_{k=1}^{\infty} \frac{n}{p^k} = \frac{n}{p-1} \leq n,$$
we have $\alpha \leq n-1$ so that $p^\alpha | b^{n-1}$. This shows that $p^\alpha \mid b^{n-1}a(a+b)(a+2b)\cdots(a+(n-1)b)$.

If $p \nmid b$, then there exists a positive integer b_1 such that $bb_1 \equiv 1 \pmod{p^\alpha}$. Note that $p \nmid b_1$. Thus
$$b_1^n a(a+b)(a+2b)\cdots(a+(n-1)b) \equiv ab_1(ab_1+1)\cdots(ab_1+n-1) \pmod{p^\alpha}.$$
As the right hand side of the above congruence is a product of n consecutive integers, it is divisible by $n!$. It is therefore divisible by p^α too. That is
$$p^\alpha \mid b_1^n a(a+b)(a+2b)\cdots(a+(n-1)b).$$
Since $p \nmid b_1$, we have $(p^\alpha, b_1^n) = 1$, so that $p^\alpha \mid a(a+b)(a+2b)\cdots(a+(n-1)b)$, and thus
$$p^\alpha \mid b^n a(a+b)(a+2b)\cdots(a+(n-1)b).$$

SINGAPORE MATHEMATICAL OLYMPIADS

2007



Roger Poh Kheng Siong
Tay Tiong Seng
Toh Tin Lam
Yang Yue

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Singapore Mathematical Olympiad (SMO) 2007

(Junior Section)

Tuesday, 29 May 2007 0930-1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter only the letters (A, B, C, D, or E) corresponding to the correct answers in the answer sheet.

For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

1. Among the four statements on integers below,

"If $a < b$ then $a^2 < b^2$";   "$a^2 > 0$ is always true";

"$-a < 0$ is always true";   "If $ac^2 < bc^2$ then $a < b$",

how many of them are correct?

(A) 0;   (B) 1;   (C) 2;   (D) 3;   (E) 4.

2. Which of the following numbers is odd for any integer values of k?

(A) $2007 + k^3$;   (B) $2007 + 7k$;   (C) $2007 + 2k^2$;   (D) $2007 + 2007k$;   (E) $2007k$.

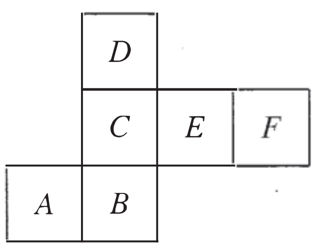
3. In a school, all 300 Secondary 3 students study either Geography, Biology or both Geography and Biology. If 80% study Geography and 50% study Biology, how many students study both Geography and biology?

(A) 30;   (B) 60;   (C) 80;   (D) 90;   (E) 150.

4. An unbiased six-sided dice is numbered 1 to 6. The dice is thrown twice and the two scores added. Which of the following events has the highest probability of occurrence?

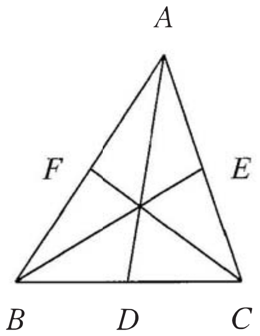
(A) The total score is a prime number; (B) The total score is a multiple of 4;
(C) The total score is a perfect square;   (D) The total score is 7;
(E) The total score is a factor of 12.

5. The cardboard below can be cut out and folded to make a cube. Which face will then be opposite the face marked A?



(A) B;   (B) C;   (C) D;   (D) E;   (E) F.

6. How many triangles can you find in the following figure?

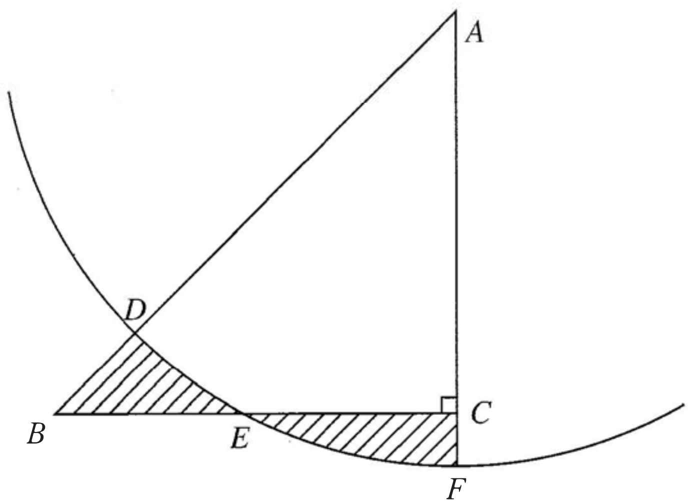


(A) 7; (B) 10; (C) 12; (D) 16; (E) 20.

7. Suppose x_1, x_2 and x_3 are roots of $(11-x)^3+(13-x)^3=(24-2x)^3$. What is the sum of $x_1+x_2+x_3$?

(A) 30; (B) 36; (C) 40; (D) 42; (E) 44.

8. In the following right-angled triangle ABC, $AC=BC=1$ and DEF is an arc of a circle with center A. Suppose the shaded areas BDE and CEF are equal and $AD=\dfrac{x}{\sqrt{\pi}}$. What is the value of x?



(A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

9. Suppose

$$\frac{1}{x} = \frac{2}{y+z} = \frac{3}{z+x} = \frac{x^2 - y - z}{x + y + z}.$$

What is the value of $\frac{z-y}{x}$?

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

10. Suppose $x^2 - 13x + 1 = 0$. What is the last digit of $x^4 + x^{-4}$?

(A) 1; (B) 3; (C) 5; (D) 7; (E) 9.

11. In a triangle ABC, it is given that $AB = 1$ cm, $BC = 2007$ cm and $AC = a$ cm, where a is an integer. Determine the value of a.

12. Find the value (in the simplest form) of $\sqrt{21 + 12\sqrt{3}} - \sqrt{21 - 12\sqrt{3}}$.

13. Find the value of

$$\frac{2007^2 + 2008^2 - 1993^2 - 1992^2}{4}.$$

14. Find the greatest integer N such that

$$N \leq \sqrt{2007^2 - 20070 + 31}.$$

15. Suppose that x and y are non-zero real numbers such that

$$\frac{x}{3} = y^2 \quad \text{and} \quad \frac{x}{9} = 9y.$$

Find the value of $x + y$.

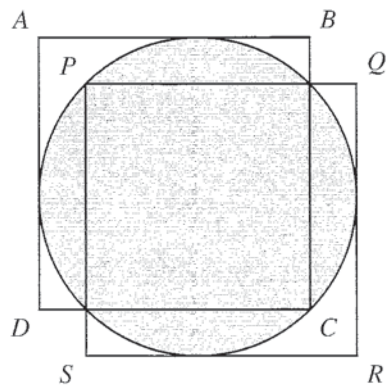
16. Evaluate the sum

$$\frac{2007}{1 \times 2} + \frac{2007}{2 \times 3} + \cdots + \frac{2007}{2006 \times 2007}.$$

17. Find the sum of the digits of the product

$$(\underbrace{111111111\ldots111}_{2007\ 1's}) \times 2007$$

18. The diagram shows two identical squares, $ABCD$ and $PQRS$, overlapping each other in such a way that their edges are parallel, and a circle of radius $(2 - \sqrt{2})$ cm covered within these squares. Find the length of the square $ABCD$.

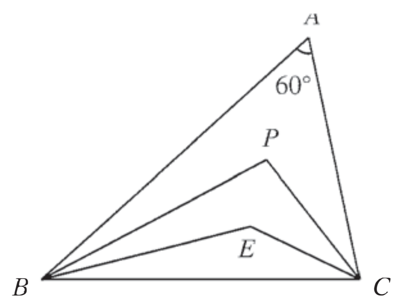


19. When 2007 bars of soap are packed into N boxes of equal size, where N is an integer strictly between 200 and 300, there are extra 5 bars remaining. Find N.

20. Suppose that $a + x^2 = 2006$, $b + x^2 = 2007$ and $c + x^2 = 2008$ and $abc = 3$. Find the value of

$$\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}.$$

21. The diagram below shows a triangle ABC in which $\angle A = 60^\circ$, BP and BE trisect $\angle ABC$; and CP and CE trisect $\angle ACB$. Let the angle $\angle BPE$ be x°. Find x.



22. Suppose that $x - y = 1$. Find the value of $x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4$.

23. How many ordered pairs of integers (m, n) where $0 < m < n < 2008$ satisfy the equation $2008^2 + m^2 = 2007^2 + n^2$?

24. If $x + \sqrt{xy} + y = 9$ and $x^2 + xy + y^2 = 27$, find the value of $x - \sqrt{xy} + y$.

25. Appending three digits at the end of 2007, one obtains an integer N of seven digits. In order to get N to be the minimal number which is divisible by 3, 5 and 7 simultaneously, what are the three digits that one would append?

26. Find the largest integer n such that $n^{6021} < 2007^{2007}$.

27. Find the value of

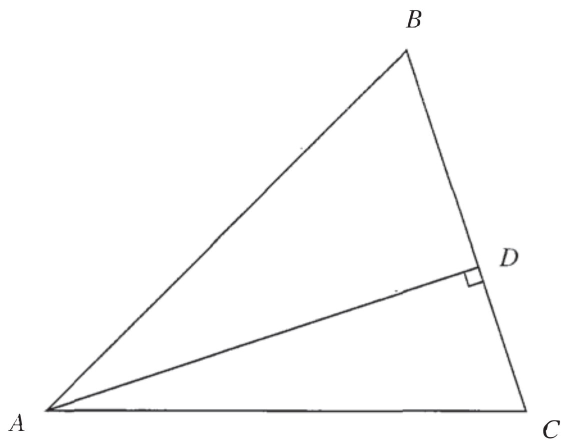
$$\frac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15}$$

when $x = \sqrt{19 - 8\sqrt{3}}$.

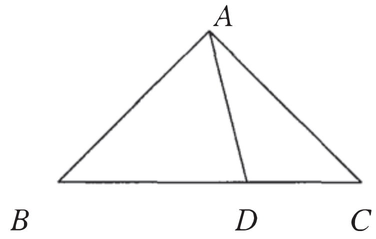
28. Find the value of a such that the two equations $x^2 + ax + 1 = 0$ and $x^2 - x - a = 0$ have one common real root.

29. Odd integers starting from 1 are grouped as follows: $(1), (3, 5), (7, 9, 11), (13, 15, 17, 19), \ldots,$ where the n-th group consists of n odd integers. How many odd integers are in the same group which 2007 belongs to?

30. In $\triangle ABC$ $\angle BAC = 45°$. D is a point on BC such that AD is perpendicular to BC. If $BD = 3$ cm and $DC = 2$ cm, and the area of the $\triangle ABC$ is x cm^2. Find the value of x.

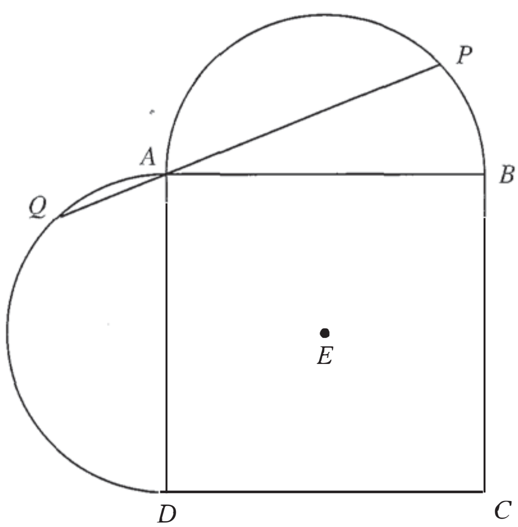


31. In $\triangle ABC$ (see below), $AB = AC = \sqrt{3}$ and D is a point on BC such that $AD = 1$. Find the value of $BD \cdot DC$.



32. Find the last digit of $2^{2^{2007}} + 1$.

33. In the following diagram, $ABCD$ is a square, and E is the center of the square $ABCD$. P is a point on a semi-circle with diameter AB. Q is a point on a semi-circle with diameter AD. Moreover, Q, A and P are collinear (that is, they are on the same line). Suppose $QA = 14$ cm, $AP = 46$ cm, and $AE = x$ cm. Find the value of x.



34. Find the smallest positive integer n such that $n(n+1)(n+2)$ is divisible by 247.

35. Find the largest integer N such that both $N + 496$ and $N + 224$ are perfect squares.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Junior Section Solutions)

1. Ans: (B)

   Only the last statement is correct: $ac^2 < bc^2$ implies $c^2 > 0$, hence $a < c$. For other statements, counterexamples can be take as $a = -2, b = -1$; $a = 0$ and $a = 0$ respectively.

2. Ans: (C)

   Because $2k^2$ is always even, thus $2007 + 2k^2$ is always odd. For other statements, either $k = 1$ or $k = 0$ gives a counterexample.

3. Ans: (D)

   Use Inclusion and Exclusion Principle.

4. Ans: (A)

   The number of occurrences for each event is: $15, 9, 7, 6$ and 12 respectively.

5. Ans: (D)

   Just imagine.

6. Ans: (D)

   Just count: Label the "center" O. There are 6 triangles like $\triangle AFO$; 3 like $\triangle AOB$; 6 like $\triangle ABD$ and 1 like $\triangle ABC$. Total: 16.

7. Ans: (B)

   Let $a = 11 - x$ and $b = 13 - x$. We have $a^3 + b^3 = (a+b)^3$. Simplify: $3ab(a+b) = 0$. Replacing a, b back in terms of x, we found the three roots are $11, 12$ and 13. Thus $x_1 + x_2 + x_3 = 36$.

8. Ans: (B)

   Since the area of sector ADF and $\triangle ABC$ are equal, we have

   $$\frac{1}{2}\left(\frac{x}{\sqrt{\pi}}\right)^2\frac{\pi}{4} = \frac{1}{2}.$$

   The result follows.

9. Ans: (B)

$$\frac{1}{x} = \frac{2}{y+z} \text{ and } \frac{1}{x} = \frac{3}{z+x}$$

tells us

$$\frac{y}{x} + \frac{z}{x} = 2 \text{ and } \frac{z}{x} + 1 = 3.$$

Thus $y = 0$ and $\frac{z}{x} = 2$. Note we didn't use the last equality, but $x = -1; y = 0$ and $z = -2$ satisfy all conditions.

10. Ans: (D)

By assumption $x + \frac{1}{x} = 13$. Thus $x^2 + \frac{1}{x^2} = 13^2 - 2 = 167$. Similarly, $x^4 + \frac{1}{x^4} = 167^2 - 2$, whose last digit is 7.

11. Ans: 2007.

Use $a < 2007 + 1$ and $2007 < a + 1$.

12. Ans: 6.

Use $21 \pm 12\sqrt{3} = (\sqrt{12} \pm 3)^2$.

13. Ans: 30000.

Using $a^2 - b^2 = (a+b)(a-b)$, $2008^2 - 1993^2 = 4001 \times 15$ and $2007^2 - 1992^2 = 3999 \times 15$. The result follows.

14. Ans: 2002.

By completion of square, $2007^2 - 20070 + 31 = (2007 - 5)^2 + 6$. The result follows.

15. Ans: 2214.

Eliminating x, we get $3y^2 = 81y$. Then $y = 27$ since $y \neq 0$. Thus $x = 2187$. The result follows.

16. Ans: 2006.

Using $\frac{1}{k \cdot (k+1)} = \frac{1}{k} - \frac{1}{k+1}$, we get

$$2007 \cdot (1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2006} - \frac{1}{2007}) = 2006.$$

17. Ans: 18063.

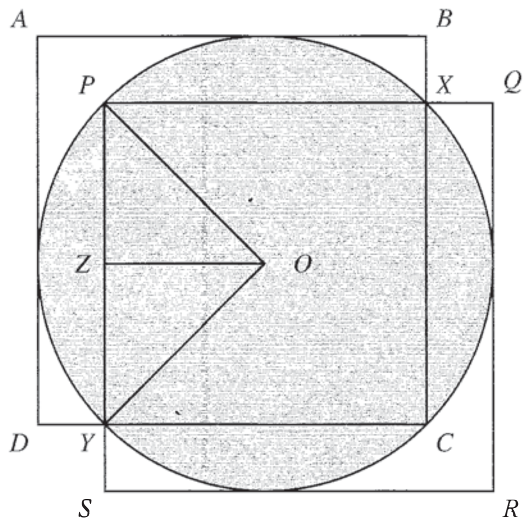
Since there is no carry, this is simply 9×2007. Or observe it is actually

$$222\underbrace{999999999\ldots999}_{2004\ 9's}777.$$

18. Ans: 1.

As the diagram below shows, $PY = \sqrt{2}OP$ and $YS = OP - \frac{1}{2}PY$. Thus,

$$PS = PY + YS = \frac{2+\sqrt{2}}{2}OP = \frac{2+\sqrt{2}}{2}(2-\sqrt{2}) = 1.$$



19. Ans: 286.

By assumption, $2007 - 5 = N \cdot k$ for some integer k. Factorize $2002 = 2 \cdot 7 \cdot 11 \cdot 13$. Since $200 < N < 300$, the only possibility is $N = 2 \cdot 11 \cdot 13 = 286$ and $k = 7$.

20. Ans: 1.

Rewrite the expression as

$$\frac{a^2 + b^2 + c^2 - bc - ca - ab}{abc} = \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2abc}.$$

Since $a - b = -1$, $b - c = -1$ and $c - a = 2$, the result follows.

21. Ans: 50.

$$\angle BPC = 180^\circ - (\angle PBC + \angle PCB) = 180^\circ - \frac{2}{3}(\angle ABC + \angle ACB) = 180^\circ - \frac{2}{3}120^\circ = 100^\circ.$$

Observe that PE bisects $\angle BPC$, the result follows.

22. Ans: 1.

We manipulate the expression and replace $x - y$ by 1 whenever necessary:

$$\begin{aligned} & x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4 \\ = \ & x^3(x-y) - y^3(x-y) - 3xy(x-y) \\ = \ & (x-y)(x^2 + xy + y^2) - 3xy \\ = \ & (x-y)^2 \\ = \ & 1. \end{aligned}$$

23. Ans: 3.

Since $2008^2 - 2007^2 = n^2 - m^2$, we have $4015 = (n+m)(n-m)$, i.e. $5 \cdot 11 \cdot 73 = (n+m)(n-m)$. There're four possibilities: $n + m = 4015, n - m = 1$; $n + m = 803, n - m = 5$; $n + m = 365, n - m = 11$ and $n + m = 73, n - m = 55$. But the first one $n = 2008$, $m = 2007$ is ruled out by assumption, the remaining pairs are $n = 404, m = 399$; $n = 188, m = 177$ and $n = 64, m = 9$.

24. Ans: 3.

Since $(x+\sqrt{xy}+y)(x-\sqrt{xy}+y) = (x+y)^2-(\sqrt{xy})^2 = x^2+xy+y^2$, we have $9\cdot(x-\sqrt{xy}+y) = 27$. The result follows.

25. Ans: 075.

It suffices to make it divisible by 105 after appending. As $2007000 = 105 \times 19114 + 30$, the least number that we need to add is 75.

26. Ans: 12.

We need the maximal n such that $(n^3)^{2007} < 2007^{2007}$. We need $n^3 < 2007$. By calculation $12^3 = 1728 < 2007 < 2197 = 13^3$. The result follows.

27. Ans: 5.

Observe $x = \sqrt{(4-\sqrt{3})^2} = 4-\sqrt{3}$ and $x^2-8x+13 = 19-8\sqrt{3}-32+8\sqrt{3}+13 = 0$. Use long division,

$$x^4-6x^3-2x^2+18x+23 = (x^2-8x+13)(x^2+2x+1)+10.$$

Thus the original expression equals

$$\frac{(x^2-8x+13)(x^2+2x+1)+10}{x^2-8x+13+2} = \frac{10}{2} = 5.$$

28. Ans: 2.

Using $ax+1 = -x-a$, we have $x = -1$ or $a = -1$. But when $a = 1$, the original equation has no real roots. Thus $x = -1$, we have $a = 2$.

29. Ans: 45.

2007 is the 1004-th odd number. If 2007 is in the group $k+1$, then $1+2+\cdots+k < 1004 \leq 1+2+\ldots(k+1)$. Thus

$$\frac{k(k+1)}{2} < 1004 \leq \frac{(k+1)(k+2)}{2}.$$

We get $k = 44$. So 2007 is in group 45 which has 45 odd integers.

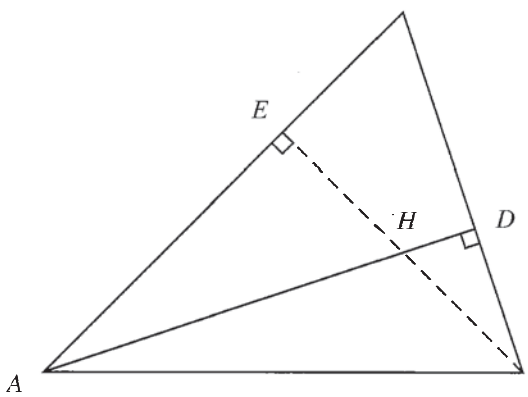
30. Ans: 15.

Construct CE which is perpendicular to AB where E is on AB. Since $\angle BAC = 45°$ (given), $AE = CE$. Thus the right $\triangle AEH$ is congruent to right $\triangle CEB$ So $AH = CB = 5$.

Next $\triangle ADB$ is similar to $\triangle CDH$, thus

$$\frac{BD}{AD} = \frac{HD}{CD} = \frac{AD-AH}{CD} \text{ which implies } \frac{3}{AD} = \frac{AD-5}{2}.$$

we get $AD^2 - 5AD - 6 = 0$ Solving $AD = 6$ only as $AD > 0$. The result follows.



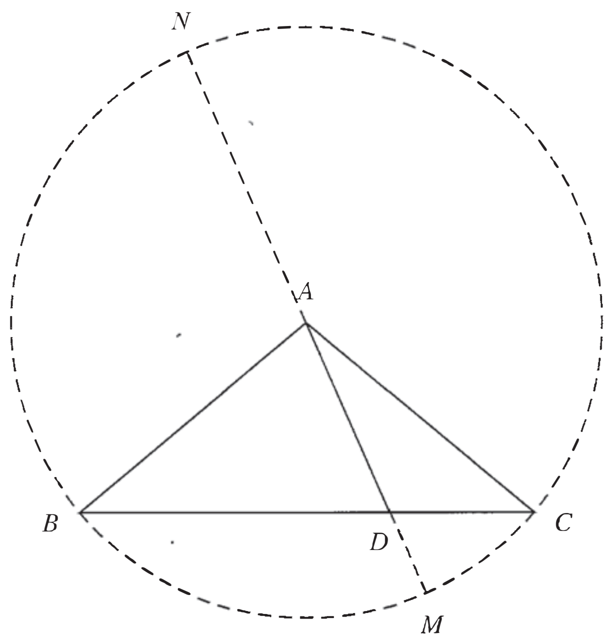
Note, if one is familiar with trigonometry, one may have an alternative solution as follows: Let α and β denote $\angle BAD$ and $\angle DAC$ respectively. Using $\tan(\alpha + \beta) = \tan 45^\circ = 1$, one get

$$\frac{\frac{3}{AD} + 2AD}{1 - \frac{3}{AD}\frac{2}{AD}} = 1,$$

consequently $AD = 6$.

31. Ans: 2.

Construct a circle with A as the center and $AB = AC = \sqrt{3}$ as the radius. Extend AD to meet the circumference at M and N as shown.



Using the Intersecting Chord Theorem

$$BD \cdot DC = MD \cdot ND = (\sqrt{3} - 1)(\sqrt{3} + 1) = 2.$$

32. Ans: 7.

Observe that for $n \geq 2$, 2^{2^n} always ends with a 6. The result follows.

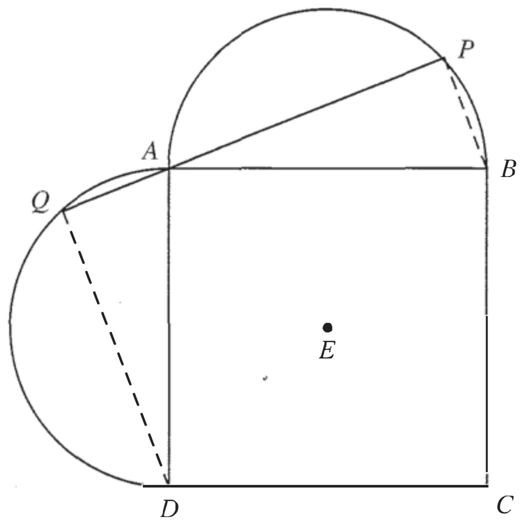
33. Ans: 34.

Join QD and PB, it is easy to show that right $\triangle DQA$ is congruent to right $\triangle APB$. Thus $PB = QA = 14$.

Apply Pythagoras' Theorem to $\triangle APB$,

$$(\sqrt{2}x)^2 = 46^2 + 14^2.$$

Solve, $x = 34$.



34. Ans: 37.

Since $247 = 13 \cdot 19$, one of $n, n+1, n+2$ is divisible by 13, call it a and one by 19 call it b. Clearly $|b - a| \leq 2$.

Let $b = 19c$. When $c = 1$, since $|b - a| \leq 2$, a is among $17, 18, 19, 20, 21$. But none is divisible by 13, hence we try $c = 2$. Now $b = 38$ and a is among $36, 37, 38, 39, 40$, hence $a = 39$. Thus the least $n = 37$.

35. Ans: 4265.

Let $N + 496 = a^2$ and $N + 224 = b^2$ for some positive integers a and b. Then $a^2 - b^2 = 496 = 224 = 272 = 2^4 \cdot 17$. Thus $17|(a+b)(a-b)$. If $17|a-b$ then $a - b \geq 17$ and $a + b \leq 16$, impossible. Thus $17|a+b$.

We have five possibilities for $(a+b, a-b)$: $(17, 16), (34, 8), (68, 4), (136, 2), (272, 1)$. Solve and discard non-integer solutions, we have $(a, b) = (21, 13), (36, 32)$ and $(69, 67)$. Thus the largest N is $69^2 - 496 = 4265$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

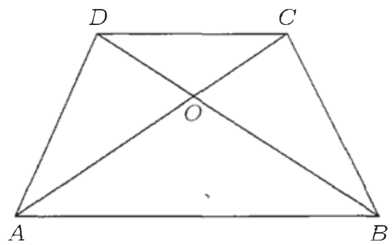
(Junior Section, Round 2)

Saturday, 30 June 2007 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. In the f llowing figure, $AB \parallel DC$, $AB = b$, $CD = a$ and $a < b$. Let S be the area of the trapezium $ABCD$. Suppose the area of $\triangle BOC$ is $2S/9$. Find the value of a/b.



2. Equilateral triangles ABE and BCF are erected externally on the sides AB and BC of a parallelogram $ABCD$. Prove that $\triangle DEF$ is equilateral.

3. Let n be a positive integer and d be the greatest common divisor of $n^2 + 1$ and $(n+1)^2 + 1$. Find all the possible values of d. Justify your answer.

4. The difference between the product and the sum of two different integers is equal to the sum of their GCD (greatest common divisor) and LCM (least common multiple). Find all these pairs of numbers. Justify your answer.

5. For any positive integer n, let $f(n)$ denote the nth positive nonsquare integer, i.e., $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, $f(4) = 6$, etc. Prove that

$$f(n) = n + \{\sqrt{n}\}$$

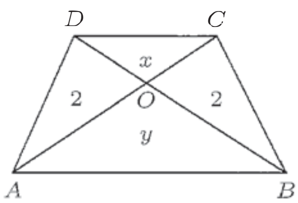
where $\{x\}$ denotes the integer closest to x. (For example, $\{\sqrt{1}\} = 1$, $\{\sqrt{2}\} = 1$, $\{\sqrt{3}\} = 2$, $\{\sqrt{4}\} = 2$.)

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Junior Section, Round 2 Solutions)

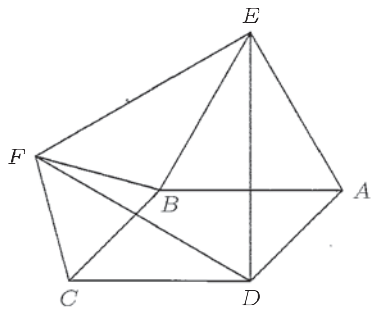
1. Without loss of generality, let $S = 9$. Then $[BOC] = 2$. Since $[ABD] = [ABC]$, we have $[AOD] = [BOC] = 2$. Let $[DOC] = x$ and $[AOB] = y$. Then $x/2 = 2/y$, i.e., $xy = 4$. Also $x + y = 5$. Thus $x(5 - x) = 4$. Solving, we get $x = 1$ and $y = 4$. Since $\triangle DOC \sim \triangle BOA$, we have $x/y = a^2/b^2$. Thus $a/b = 1/2$.



2. We have

$$\begin{aligned}\angle EBF &= 240^\circ - \angle ABC = 240^\circ - (180^\circ - \angle BCD) \\ &= 60^\circ + \angle BCD = \angle DCF\end{aligned}$$

Also $FB = FC$ and $BE = BA = CD$. Thus $\triangle FBE \cong \triangle FCD$. Therefore $FE = FD$. Similarly $\triangle EAD \cong \triangle DCF$. Therefore $ED = DF$. Thus $\triangle DEF$ is equilateral.



3. For $n = 1$ and $n = 2$, the gcd are 1 and 5, respectively. Any common divisor d of $n^2 + 1$ and $(n + 1)^2 + 1$ divides their difference, $2n + 1$. Hence d divides $4(n^2 + 1) - (2n + 1)(2n - 1) = 5$. Thus the possible values are 1 and 5.

4. Let the integers be x and y and assume that $x > y$. First we note that x and y are both nonzero and that their GCD and LCM are both positive by definition. Let M be the GCD, then $|x| = Ma$ and $|y| = Mb$, where a and b are coprime integers. Thus the LCM of x and y is Mab. If $y = 1$, then $M = 1$ and it's easily checked that there is no solution. When $y > 1$, $xy > x + y$. Thus $xy - (x + y) = M + Mab$. After substituting for x and y and simplifying, we have

$$ab(M-1) = 1 + a + b \quad \Rightarrow \quad M = 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} \quad \Rightarrow \quad 1 < M \leq 4.$$

If $M = 2$, then $ab - a - b = 1$, i.e., $(a-1)(b-1) = 2$. Thus $a = 3$, $b = 2$ or $x = 6$, $y = 4$. Similarly, when $M = 3$, we get $2ab - a - b = 1$. Multiplying throughout by 2 and then factorize, we get $(2a-1)(2b-1) = 3$ which gives $x = 6$ and $y = 3$. When $M = 4$, we get $x = y = 4$ which is rejected as x and y are distinct.
Next we consider the case $x > 0 > y$. Then $x = Ma$ and $y = -Mb$. Using similar arguments, we get $x + y - xy = M + Mab$. Thus $M = 1 + \frac{1}{ab} + \frac{1}{a} - \frac{1}{b}$ which yields $1 \leq M \leq 2$. When $M = 1$, we get $a = 1 + b$. Thus the solutions are $b = t$, $a = 1 + t$ or $x = 1 + t$, $y = -t$, where $t \in \mathbb{N}$. When $M = 2$, the equation simplifies to $(a-1)(b+1) = 0$. Thus we get $a = 1$ and b arbitrary as the only solution. The solutions are $x = 2$, $y = -2t$, where $t \in \mathbb{N}$.
Finally, we consider the case $0 > x > y$. Here $x = -Ma$, $y = -Mb$ and $M^2ab + Ma + Mb = M + Mab$. Since $M2ab \geq Mab$ and $Ma + Mb > M$, there is no solution.
Thus the solutions are $(6,3)$, $(6,4)$, $(1+t, -t)$ and $(2, -2t)$ where $t \in \mathbb{N}$.

5. If $f(n) = n + k$, then there are exactly k square numbers less than $f(n)$. Thus $k^2 < f(n) < (k+1)^2$. Now we show that $k = \{\sqrt{n}\}$. We have

$$k^2 + 1 \leq f(n) = n + k \leq (k+1)^2 - 1.$$

Hence

$$\left(k - \frac{1}{2}\right)^2 + \frac{3}{4} = k^2 - k + 1 \leq n \leq k^2 + k = \left(k + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Therefore

$$k - \frac{1}{2} < \sqrt{n} < k + \frac{1}{2},$$

so that $\{\sqrt{n}\} = k$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Senior Section)

Tuesday, 29 May 2007 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answers in the answer sheet by shading the bubbles containing the letters (A, B, C, D *or* E) *corresponding to the correct answers.*

For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Find the sum of the digits of the product $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\ldots\left(1+\frac{1}{2006}\right)\left(1+\frac{1}{2007}\right)$.

   (A) 5
   (B) 6
   (C) 9
   (D) 10
   (E) 13

2. A bag contains x green and y red sweets. A sweet is selected at random from the bag and its colour noted. It is then replaced into the bag together with 10 additional sweets of the same colour. A second sweet is next randomly drawn. Find the probability that the second sweet is red.

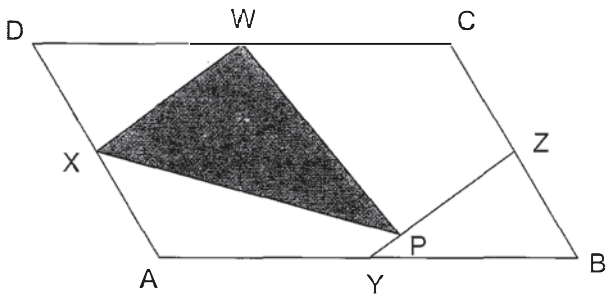
   (A) $\frac{y+10}{x+y+10}$
   (B) $\frac{y}{x+y+10}$
   (C) $\frac{y}{x+y}$
   (D) $\frac{x}{x+y}$
   (E) $\frac{x+y}{x+y+10}$

3. What is the remainder when the number

$$(\underbrace{999\,999\,999\ldots999}_{2008\ 9\text{'s}})^{2007}-(\underbrace{333\,333\,333\ldots333}_{2008\ 3\text{'s}})^{2007}$$

   is divided by 11?
   (A) 0
   (B) 2
   (C) 4
   (D) 6
   (E) None of the above

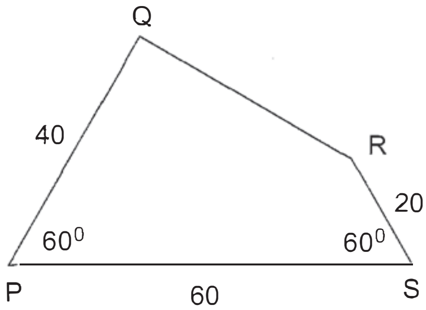
4. W, X, Y and Z are the midpoints of the four sides of parallelogram ABCD. P is a point on the line segment YZ. What percent of the area of parallelogram ABCD is triangle PXW?



(A) 50%
(B) 45%
(C) 30%
(D) 25%
(E) 20%

5. Four rods are connected together with flexible joints at their ends to make a quadrilateral as shown. Rods $PQ = 40$ cm, $RS = 20$ cm, $PS = 60$ cm and $\angle QPS = \angle RSP = 60^{\circ}$. Find $\angle QRS$.

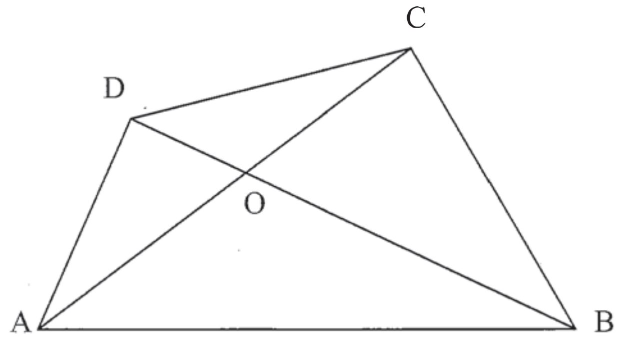
(A) 100°
(B) 105°
(C) 120°
(D) 135°
(E) 150°



6. When 2007 bars of soap are packed into N boxes, where N is a positive integer, there is a remainder of 5. How many possible values of N are there?
(A) 14
(B) 16
(C) 18
(D) 20
(E) 13

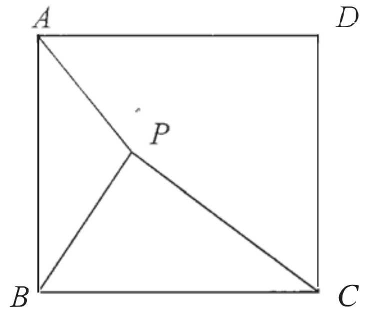
7. Suppose a_n denotes the last two digits of 7^n. For example, $a_2 = 49$, $a_3 = 43$. Find the value of $a_1 + a_2 + a_3 + \dots\dots\dots + a_{2007}$
(A) 50189
(B) 50199
(C) 50209
(D) 50219
(E) 50229

8. The diagram below shows a quadrilateral ABCD where AB = 10, BC = 6, CD = 8 and DA = 2. The diagonals AC and BD intersect at the point O and that $\angle COB = 45^0$. Find the area of the quadrilateral ABCD.



(A) 28
(B) 29
(C) 30
(D) 31
(E) 32

9. In the following diagram, $ABCD$ is a square with $PA = a$, $PB = 2a$ and $PC = 3a$. Find $\angle APB$.



(A) 120^0
(B) 130^0
(C) 135^0
(D) 140^0
(E) 145^0

10. What is the largest possible prime value of $n^2 - 12n + 27$, where n ranges over all positive integers?
(A) 91
(B) 37
(C) 23
(D) 17
(E) 7

Short Questions

11. Suppose that $\log_2[\log_3(\log_4 a)] = \log_3[\log_4(\log_2 b)] = \log_4[\log_2(\log_3 c)] = 0$. Find the value of $a + b + c$.

12. Find the unit digit of $17^{17} \times 19^{19} \times 23^{23}$.

13. Given that $x + y = 12$ and $xy = 50$, find the exact value of $x^2 + y^2$.

14. Suppose that $(21.4)^a = (0.00214)^b = 100$. Find the value of $\frac{1}{a} - \frac{1}{b}$.

15. Find the value of $100(\sin 253^\circ \sin 313^\circ + \sin 163^\circ \sin 223^\circ)$

16. The letters of the word MATHEMATICS are rearranged in such a way that the first four letters of the arrangement are all vowels. Find the total number of distinct arrangements that can be formed in this way.
(Note: The vowels of English language are A, E, I, O, U)

17. Given a set $S = \{1, 2, 3, \ldots, 199, 200\}$. The subset $A = \{a, b, c\}$ of S is said to be "*nice*" if $a + c = 2b$. How many "*nice*" subsets does S have?
(Note: The order of the elements inside the set does not matter. For example, we consider $\{a, b, c\}$ or $\{a, c, b\}$ or $\{c, b, a\}$ to be the same set.)

18. Find the remainder when $2^{55} + 1$ is divided by 33.

19. Given that the difference between two 2-digit numbers is 58 and these last two digits of the squares of these two numbers are the same, find the smaller number.

20. Evaluate $256 \sin 10^0 \sin 30^0 \sin 50^0 \sin 70^0$.

21. Find the greatest integer less than or equal to $\left(2 + \sqrt{3}\right)^3$.

22. Suppose that x_1, x_2 and x_3 are the three roots of $(11 - x)^3 + (13 - x)^3 = (24 - 2x)^3$. Find the value of $x_1 + x_2 + x_3$.

23. In ΔABC, $\angle CAB = 30^\circ$ and $\angle ABC = 80^\circ$. The point M lies inside the triangle such that $\angle MAC = 10^\circ$ and $\angle MCA = 30^\circ$. Find $\angle BMC$ in degrees.

24. How many positive integer n less than 2007 can we find such that $\left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \left[\frac{n}{6}\right] = n$ where $[x]$ is the greatest integer less than or equal to x? (For example, [2.5] = 2; [5] = 5; [-2.5] = -3 etc.)

25. In ΔABC, let $AB = c$, $BC = a$ and $AC = b$. Suppose that $\frac{b}{c-a} - \frac{a}{b+c} = 1$, find the value of the greatest angle of ΔABC in degrees.

26. Find the number of integers N satisfying the following two conditions:
    (i) $1 \le N \le 2007$; and
    (ii) either N is divisible by 10 or 12 (or both).

27. Suppose a and b are the roots of $x^2 + x \sin\alpha + 1 = 0$ while c and d are the roots of the equation $x^2 + x\cos\alpha - 1 = 0$. Find the value of $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}$.

28. A sequence $\{a_n\}$ is defined by $a_1 = 2$, $a_n = \frac{1 + a_{n-1}}{1 - a_{n-1}}$, $n \ge 2$. Find the value of $-2008\, a_{2007}$.

29. Let x, y and z be three real numbers such that $xy + yz + xz = 4$. Find the least possible value of $x^2 + y^2 + z^2$.

30. P is the set $\{1, 2, 3, \ldots, 14, 15\}$. If $A = \{a_1, a_2, a_3\}$ is a subset of P where $a_1 < a_2 < a_3$ such that $a_1 + 6 \le a_2 + 3 \le a_3$. How many such subsets are there of P ?

31. It is given that x and y are two real numbers such that

$$(x+y)^4+(x-y)^4=4112 \text{ and } x^2-y^2=16.$$

Find the value of x^2+y^2.

32. Let A be an angle such that $\tan 8A=\dfrac{\cos A-\sin A}{\cos A+\sin A}$. Suppose $A=x^{\circ}$ for some positive real number x. Find the smallest possible value of x.

33. Find the minimum value of $\sum_{k=1}^{100}|n-k|$, where n ranges over all positive integers.

34. Find the number of pairs of positive integers (x, y) are there which satisfy the equation $2x + 3y = 2007$.

35. If $S=\dfrac{1}{1+1^2+1^4}+\dfrac{2}{1+2^2+2^4}+\dfrac{3}{1+3^2+3^4}+\dots\dots\dots+\dfrac{200}{1+200^2+200^4}$, find the value of $80402\times S$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Senior Section Solution)

1. (A)

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\ldots\left(1+\frac{1}{2006}\right)\left(1+\frac{1}{2007}\right)=\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\ldots\frac{2008}{2007}=\frac{2008}{2}=1004,$$

so the sum of the digits is 5.

2. (C)

Probability $=\left(\frac{x}{x+y}\times\frac{y}{x+y+10}\right)+\left(\frac{y}{x+y}\times\frac{y+10}{x+y+10}\right)$, which upon simplification yields (C) as the answer.

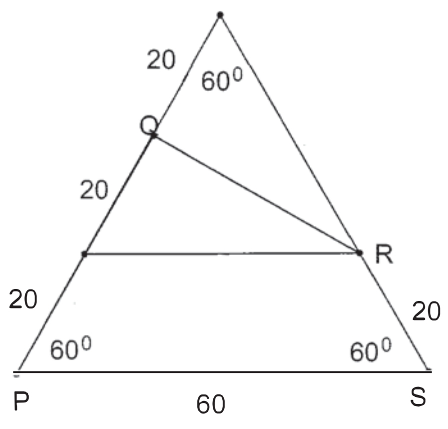
3. (A)

Observe that each of the two numbers is divisible by 11, hence the difference is also divisible by 11. Hence the remainder is zero.

4. (D)

Parallelogram WXYZ is half of ABCD. Triangle PXW is half of WXYZ. Thus, PXW is a quarter of ABCD. Hence (D)

5. (E)



We extend the given figure to the above, into an equilateral triangle. It is not possible to show that $\angle PQR = 90^{\circ}$. Hence, $\angle QRS = 30^{\circ} + 120^{\circ}$.

6. (A)

2007 – 5 = 2002. N is a factor of 2002 and 2002 = 2 x 7 x 11 x 13.

There are altogether 2x2x2x2 = 16 factors of N. However, N must exceed 5. So, N cannot be 1 or 2. Hence there are 14 possible choices of N.

7. (B)

Observe that a_n repeats itself as shown in the following table.

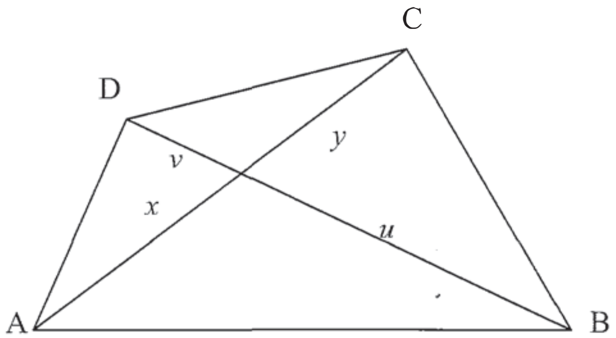
n	1	2	3	4	5	6	7	....
a_n	07	49	43	01	07	49	43	....

Further, $a_1 + a_2 + a_3 + a_4 = 100$.

Notice that 2007 = 4 x 501 + 3.

Hence the sum equals 501 x 100 + 7 + 49 + 43 = 50199.

8. (D)



Area of quadrilateral = $\frac{1}{2}(xu + yu + yv + xv)\sin 45^0$.

Using Cosine Rule, we have

$$x^2 + u^2 + 2xu\cos 45^0 = 10^2 \quad (1)$$
$$u^2 + y^2 - 2uy\cos 45^0 = 6^2 \quad (2)$$
$$y^2 + v^2 + 2yv\cos 45^0 = 8^2 \quad (3)$$
$$x^2 + v^2 - 2xv\cos 45^0 = 2^2 \quad (4)$$

(1) – (2) + (3) – (4):

$$2(xu + yu + yv + vx)\cos 45^0 = 10^2 - 6^2 + 8^2 - 2^2$$

Hence area of quadrilateral = $\dfrac{100 - 36 + 64 - 4}{4} = 31$.

9. (C)

Let $AB = x$ and $\angle PAB = \alpha$. By using Cosine Rule and Pythagoras' Theorem we have

$$x^2 + a^2 - 2ax\cos\alpha = (2a)^2$$
$$(x - a\cos\alpha)^2 + (x - a\sin\alpha)^2 = (3a)^2.$$

We obtain from the above two equations that

$$\cos\alpha = \frac{x^2-3a^2}{2ax} \quad \text{and} \quad \sin\alpha = \frac{x^2-5a^2}{2ax}.$$

Using $\sin^2\alpha + \cos^2\alpha = 1$, we have

$$\left(x^2-3a^2\right)^2 + \left(x^2-5a^2\right)^2 = 4a^2x^2$$

Therefore $x^4 - 10a^2x^2 + 17a^4 = 0$.

Solving, we have $x^2 = \left(5 \pm 2\sqrt{2}\right)a^2$, but since $x > a$, we have $x^2 = \left(5+2\sqrt{2}\right)a^2$.

Hence $\cos \angle APB = -\frac{\sqrt{2}}{2}$, so the required angle is 135^0.

10. (E)

Note that $n^2 - 12n + 27 = (n-9)(n-3)$. For this number to be a prime, either $n =$ 10, in which case $n^2 - 12n + 27 = 7$; or $n = 2$, in which case $n^2 - 12n + 27 = 7$. Since a prime number cannot be factorized in other ways, we know that 7 is the only answer.

11. Answer: 89

$\log_2 [\log_3 (\log_4 a)] = 0$ implies $\log_3 (\log_4 a) = 1$. Hence $\log_4 a = 3$, hence $a = 64$. Similarly, $b = 16$ and $c = 9$. Therefore $a + b + c = 89$.

12. Answer: 1

$17^{17} \equiv 7^{17} \equiv 7 \pmod{10}$

$19^{19} \equiv 9^{19} \equiv 9 \pmod{10}$

$23^{23} \equiv 3^{23} \equiv 7 \pmod{10}$

Since $7 \times 9 \times 7 = 441 \equiv 1 \pmod{10}$, the unit digit is 1.

13. Answer: 44

Use the identity $x^2 + y^2 = (x+y)^2 - 2xy = 144 - 100 = 44$.

(Note: In this case, one may not be able to obtain the answer by guess-and-check; the values of x and y are not even real numbers)

14. Answer: 2

We have $a = \frac{2}{\log_{10} 21.4}$ and $b = \frac{2}{(\log_{10} 21.4) - 4}$. Hence direct computation yields $\frac{1}{a} - \frac{1}{b} = 2$.

15. Answer: 50

$$\begin{aligned}
&100(\sin 253^\circ \sin 313^\circ + \sin 163^\circ \sin 223^\circ) \\
&\quad = 100(\sin 73^\circ \sin 47^\circ - \sin 17^\circ \sin 43^\circ) \\
&\quad = 100(\sin 47^\circ \cos 17^\circ - \cos 47^\circ \sin 17^\circ) \\
&\quad = 100 \sin(47^\circ - 17^\circ) = 100 \sin 30^\circ \\
&\quad = 50
\end{aligned}$$

16. Answer: 15 120

Total number of rearrangements = $\frac{4!}{2!} \times \frac{7!}{2!2!} = 12 \times 1260 = 15120.$

17. Answer: 9900
Since $a + c$ is even, a and c have the same parity (either both odd or both even). There are 100 odd and 100 even numbers in S. Once a and c are chosen, b is determined. The number of "*nice*" subsets is $2\binom{100}{2} = 100(99) = 9900$

18: Answer: 0
Note that $x^{11} + 1$ is divisible by $x + 1$ by Factor Theorem. Hence $2^{55} + 1 = 32^{11} + 1$ is divisible by $32 + 1 = 33$, i.e. $2^{55} + 1$ is divisible by 33.

19. Answer: 21
Let the numbers be n and m where $n > m$.
$n - m = 58$
$n^2 - m^2 = (n - m)(n + m) = 58(n + m)$ is a multiple of 100, that is,
$58(n + m) = 100k$ for some integer k. This simplifies to
$29(n + m) = 50k$
Since gcd(29, 50) = 1, we must have $n + m = 50p$ for some positive integer p.
Solving the above simultaneously with $n - m = 58$ and bearing in mind that both m and n are two digit numbers, only $n = 79$ and $m = 21$ satisfy the question.
Thus the smaller number is 21.

20. Answer: 16
$256 \sin 10^0 \sin 30^0 \sin 50^0 \sin 70^0$

$$= \frac{2 \times 256 \sin 10^0 \cos 10^0 \sin 30^0 \cos 40^0 \cos 20^0}{2 \cos 10^0}$$

$$= \frac{128 \sin 20^0 \cos 20^0 \cos 40^0}{2 \cos 10^0}$$

$$= \frac{64 \sin 40^0 \cos 40^0}{2 \cos 10^0}$$

$$= \frac{32 \sin 80^0}{2 \cos 10^0}$$

$= 16.$

21. Answer: 51
Using binomial theorem it is easy to see that

$$\left(2 + \sqrt{3}\right)^3 + \left(2 - \sqrt{3}\right)^3 = 52,$$

and that $0 < 2 - \sqrt{3} < 1$, or $0 < \left(2 - \sqrt{3}\right)^3 < 1$ so that we have

$51 < \left(2 + \sqrt{3}\right)^3 < 52.$

22. Answer: 36

Let $a = 11 - x$ and $b = 13 - x$. Hence the equation becomes

$$a^3 + b^3 = (a + b)^3 \qquad (1)$$

Using the binomial expansion

$$(a + b)^3 = a^3 + b^3 + 3ab\,(a + b),$$

Equation (1) becomes

$$ab\,(a + b) = 0 \qquad (2)$$

Thus the solution of (1) becomes

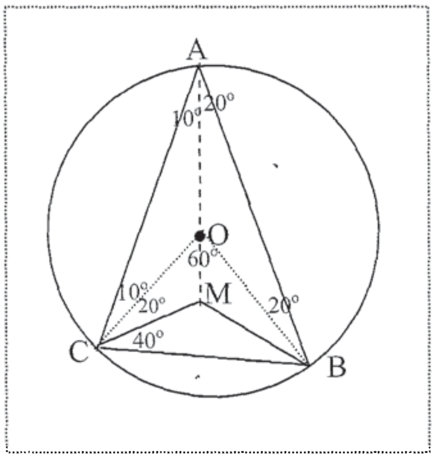
$a = 0$, $b = 0$ or $a + b = 0$

or equivalently, the solution of the original equation becomes $x = 11$, 12 or 13.

Hence $x_1 + x_2 + x_3 = 11 + 12 + 13 = 36$.

23. Answer: 110^0.

Construct a circumcircle of the triangle ABC, with O as the centre as shown.



Note that $\angle ACB = 70^o$

Since OC = OB and $\angle COB = 60^o \Rightarrow \angle OCB = 60^o$

ΔCOB is an equilateral triangle.

Thus $\angle OCA = 70^o - 60^o = 10^o = \angle OAC$

But $\angle MAC = 10^o$ (given). So AOM lies on a straight line.

$\angle AOC = 160^o \Rightarrow \angle COM = 20^o$.

Since $\angle MCA = 30^o$ (given) and $\angle OCA = 10^o$, thus $\angle MCO = 20^o$. That means ΔMCO is an isosceles triangle and $\angle MCB = 70 - 30 = 40^o$

So BM is the perpendicular bisector of the equilateral triangle OBC. Thus

$\angle OBM = 60^o \div 2 = 30^o$

$\angle BMC = 180^o - 40^o - 30^o = 110^o$

24. Answer: 334

$$\left[\frac{n}{2}\right] \le \frac{n}{2},\ \left[\frac{n}{3}\right] \le \frac{n}{3},\ \left[\frac{n}{6}\right] \le \frac{n}{6}$$

Given $\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\left[\frac{n}{6}\right]=n$ but we have $\frac{n}{2}+\frac{n}{3}+\frac{n}{6}=n$

So $\left[\frac{n}{2}\right]=\frac{n}{2}$, $\left[\frac{n}{3}\right]=\frac{n}{3}$, $\left[\frac{n}{6}\right]=\frac{n}{6}$

Thus n must be a multiple of 6

There are $\left[\frac{2007}{6}\right]=334$ of them.

25. Answer: 120^0.

It is given that $\frac{b}{c-a}-\frac{a}{b+c}=1$. This can be rearranged into $b^2+a^2-c^2=-ab$.

By using cosine rule for triangle, $\cos C=\frac{b^2+a^2-c^2}{2ab}=-\frac{1}{2}$. Hence $C=120^0$ and must be the greatest angle.

26. Answer: 334

Number of integers divisible by 10 = $\left[\frac{2007}{10}\right]=200$.

Number of integers divisible by 12 = $\left[\frac{2007}{12}\right]=167$

Number of integers divisible by both 12 and 10 = $\left[\frac{2007}{60}\right]=33$.

By the principle of inclusion and exclusion, the number of integers divisible by either 10 or 12 (or both) = 200 + 167 – 33 = 334.

27. Answer: 1

We have $ab=1$. Hence $a^2=\frac{1}{b^2}$ and $b^2=\frac{1}{a^2}$.

Also, $cd=-1$. Hence $d^2=\frac{1}{c^2}$ and $c^2=\frac{1}{d^2}$.

Hence
$$\begin{aligned}\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2} &= a^2+b^2+c^2+d^2\\ &= (a+b)^2-2ab+(c+d)^2-2cd\\ &= \sin^2\alpha-2+\cos^2\alpha+2\\ &= 1\end{aligned}$$

28. Answer: 1004

Since $a_1 = 2$, $a_2 = -3$, $a_3 = -\frac{1}{2}$, $a_4 = \frac{1}{3}$, $a_5 = 2$, thus we consider

$$a_{n+4} = \frac{1+a_{n+3}}{1-a_{n+3}} = \frac{1+\frac{1+a_{n+2}}{1-a_{n+2}}}{1-\frac{1+a_{n+2}}{1-a_{n+2}}} = -\frac{1}{a_{n+2}} = -\left(\frac{1-a_{n+1}}{1+a_{n+1}}\right)$$

$$= -\frac{1-\frac{1+a_n}{1-a_n}}{1+\frac{1+a_n}{1-a_n}} = a_n$$

$a_{2007} = a_{4\times501+3} = a_3 = -\frac{1}{2}$

$-2008\, a_{2007} = 1004$

29. Answer: 4

Use the identity

$x^2 + y^2 + z^2 - (xy + yz + xz) = \frac{1}{2}\left((x-y)^2 + (y-z)^2 + (x-z)^2\right) \geq 0$, the answer follows immediately.

30. Answer: 165

Supposing there are x_1 numbers smaller than a_1, x_2 numbers between a_1 and a_2 , x_3 numbers between a_2 and a_3 and x_4 numbers greater than a_3 .

Finding the number of possible subsets of A is equivalent to finding the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 12$ with the conditions that $x_1 \geq 0$, $x_2 \geq 2$, $x_3 \geq 2$ and $x_4 \geq 0$.

The number of solution of this latter equation is equivalent to the number of solutions of the equation $y_1 + y_2 + y_3 + y_4 = 8$, where y_1, y_2, y_3 and y_4 are nonnegative integers.

Hence the answer is $\binom{11}{3} = 165$.

31. Answer: 34

By binomial theorem, one sees that

$$(x+y)^4 + (x-y)^4 = 2(x^4 + 6x^2y^2 + y^4)$$

Thus the first equation becomes

$$2(x^4 + 6x^2y^2 + y^4) = 4112$$

$$x^4 + 6x^2y^2 + y^4 = 2056$$

$$(x^2 - y^2)^2 + 8x^2y^2 = 2056$$

$$(16)^2 + 8x^2y^2 = 2056$$

$$x^2y^2 = 2256$$

Therefore,

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 1156$$

Hence $x^2 + y^2 = \sqrt{1156} = 34$.

32. Answer: 5

Since $\dfrac{\sin 8A}{\cos 8A} = \tan 8A = \dfrac{\cos A - \sin A}{\cos A + \sin A}$, by rearranging we obtain

$$\sin 8A\,(\cos A + \sin A) = \cos 8A\,(\cos A - \sin A)$$

$$\sin 8A \cos A + \cos 8A \sin A = \cos 8A \cos A - \sin 8A \sin A$$

$$\sin (8A + A) = \cos (8A + A)$$

$$\sin 9A = \cos 9A$$

which reduces to

$$\tan 9A = 1$$

The smallest possible of $9A = 45^0$, which gives $x = 5$.

33. Answer: 2500

It is given that n is a positive integer.

For $n \geq 100$, $\sum_{k=1}^{100} |n-k| = 100n - 5050$, so that its minimum value is 4950 which occurs at $n = 100$.

For $n < 100$, $\sum_{k=1}^{100} |n-k| = n^2 - 101n + 5050$. If n is a positive integer, its minimum value can occur at either $n = 50$ or $n = 51$ only. By direct checking, its minimum value is 2500.

34. Answer: 334

$2x + 3y = 2007 \Rightarrow 2x = 2007 - 3y = 3(669 - y)$.

Since 2 and 3 are relatively prime, it follows that x is divisible by 3. Write $x = 3t$, where t is a positive integer.

The equation reduces to $y = 669 - 2t$.

Since $669 - 2t > 0$, we have $t < 334.5$, and since t is a positive integer, we have $1 \leq t \leq 334$.

Conversely, for any positive integer t satisfying $1 \leq t \leq 334$, it is easily seen that $(3t, 669 - 2t)$ is a pair of positive integers which satisfy the given equation.

Therefore there are 334 pairs of positive integers satisfying the given equations.

35. Answer: 40 200

Note that $\dfrac{k}{1+k^2+k^4} = \dfrac{1}{2}\left[\dfrac{1}{k(k-1)+1} - \dfrac{1}{k(k+1)+1}\right]$ for all integers k.

Hence the required sum can be found by the method of difference

$$= \frac{1}{2}\left(\begin{array}{l}\dfrac{1}{1\times 0+1} - \dfrac{1}{1\times 2+1} + \dfrac{1}{2\times 1+1} - \dfrac{1}{2\times 3+1} + \dfrac{1}{3\times 2+1} - \dfrac{1}{3\times 4+1} + \ldots\ldots + \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \dfrac{1}{200\times 199+1} - \dfrac{1}{200\times 201+1}\end{array}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{40201}\right)$$

$$= \frac{20100}{40201}$$

Hence 80402 x S = 40 200.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Senior Section, Round 2)

Saturday, 30 June 2007 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. It is given that x, y, z are 3 real numbers such that

$$\frac{x-y}{2+xy}+\frac{y-z}{2+yz}+\frac{z-x}{2+zx}=0.$$

Is it true that at least two of the three numbers must be equal? Justify your answer.

2. For any positive integer n, let $f(n)$ denote the nth positive nonsquare integer, i.e., $f(1)=2$, $f(2)=3$, $f(3)=5$, $f(4)=6$, etc. Prove that

$$f(n)=n+\{\sqrt{n}\}$$

where $\{x\}$ denotes the integer closest to x. (For example, $\{\sqrt{1}\}=1$, $\{\sqrt{2}\}=1$, $\{\sqrt{3}\}=2$, $\{\sqrt{4}\}=2$.)

3. In the equilateral triangle ABC, M, N are the midpoints of the sides AB, AC, respectively. The line MN intersects the circumcircle of $\triangle ABC$ at K and L and the lines CK and CL meet the line AB at P and Q, respectively. Prove that $PA^2\cdot QB=QA^2\cdot PB$.

4. Thirty two pairs of identical twins are lined up in an 8×8 formation. Prove that it is possible to choose 32 persons, one from each pair of twins, so that there is at least one chosen person in each row and in each column.

5. Find the maximum and minimum of $x+y$ such that

$$x+y=\sqrt{2x-1}+\sqrt{4y+3}.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Senior Section, Round 2 Solutions)

1. Yes. Multiplying both sides by $(2+xy)(2+yz)(2+zx)$, we get

$$F := (x-y)(2+yz)(2+zx)+(y-z)(2+xy)(2+zx)+(z-x)(2+zx)(2+xy)=0.$$

Now regard F as a polynomial in x. Since $F=0$ when $x=y$, $x-y$ is a factor of F. Similarly, $y-z$ and $z-x$ are also factors of F. Since F is of degree 3,

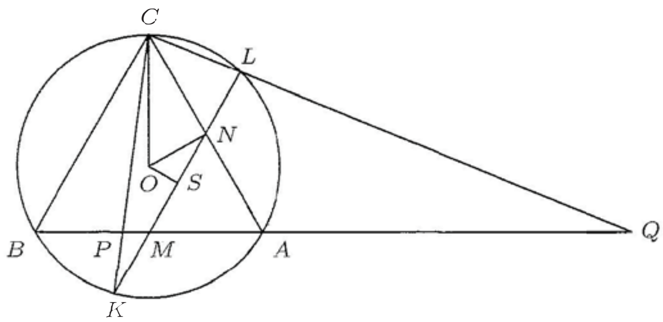
$$F \equiv k(x-y)(y-z)(z-x)$$

for some constant k. By letting $x=1$, $y=-1$, $z=0$, we have $k=2$. Thus

$$F \equiv 2(x-y)(y-z)(z-x)$$

2. See Junior Section Round 2, Question 5.

3. Let the radius of the circumcircle be $R=4t$ and the centre be O. Let S be the foot of the perpendicular from O to MN. Then $OS=ON/2=t$, $KS=\sqrt{16t^2-t^2}=\sqrt{15}t$, $MS=\sqrt{4t^2-t^2}=\sqrt{3}t$. Thus $KM=KS-MS=(\sqrt{15}-\sqrt{3})t$. Since $\triangle PMK \sim \triangle PBC$ and $BC=\sqrt{24}t$, we have $PM/PB=MK/BC=(\sqrt{5}-1)/4$, i.e., $PM=(\sqrt{5}-1)PB/4$. Thus $PA=PM+MA=PM+BM=2PM+PB=(\sqrt{5}+1)PB/2$ and $BA=PA+PB=(\sqrt{5}+3)PB/2$. Therefore $PA^2=PB\cdot BA$. Similarly, $QA^2=QB\cdot BA$. Thus $PA^2\cdot QB=QA^2\cdot PB$.



Second solution: Note that $\angle BCL = \angle CBK$ and $\angle BKC = 60^\circ$. Thus $\angle BCL + \angle BCP = \angle CBK + \angle BCK = 180^\circ - \angle BKC = 120^\circ$. This implies that $\angle BQC = 180^\circ - \angle CBQ - \angle BCQ = 120^\circ - \angle BCL = \angle BCP$, and therefore $\triangle BCQ \sim \triangle BPC$. Thus, $CQ/PC = BQ/BC = BC/BP$, i.e., $\left(\frac{CQ}{PC}\right)^2 = \frac{BQ}{BC} \cdot \frac{BC}{BP} = \frac{BQ}{BP}$. Now since $KL \parallel BC$, A is also midpoint along the arc of KL. Thus, $\angle KCA = \angle ACL$. By the angle bisector theorem on $\triangle PCQ$, we have $CQ/CP = AQ/AP$. Combining, we get $\left(\frac{AQ}{AP}\right)^2 = \left(\frac{CQ}{PC}\right)^2 = \frac{BQ}{BP}$, i.e., $PA^2 \cdot QB = QA^2 \cdot QB$.

4. Suppose on the contrary that this is not possible. We say that a row or a column is *covered* if it contains one of the chosen persons. Choose 32 persons so that the total number of covered rows and columns is maximum. Without loss of generality, assume that column 1 is not covered. Then no pair of twins can be in column 1. The counterparts of these 8 persons must be chosen. By the maximality condition, these 8 persons must be the only chosen persons either in their row or column. If a row contains only 1 chosen person, then there are 6 other persons (excluding those in column 1) in that row who are not chosen. Since 32 persons are not chosen, we have at most 4 such rows. Similarly, there are at most 3 such columns and we have a contradiction because we have 8 such rows or columns.

   2nd solution: We'll show that at most 10 persons are needed. Note that if a set of people cover all the rows and columns, then they will still have the same property if two rows or two columns are interchanged. Thus if the configuration obtained by interchanging some pairs of rows and some pairs of columns can be covered by a set of people, then the same set of people will cover the original configuration. We also let M_i denote the configuration obtained by deleting the first i rows and columns of the original configuration. Let us denote a person by (a, b) if he is in the a-th column b-th row. Choose $(1, 1)$. In M_1, there is a person who is not the counterpart of $(1, 1)$. Without loss of generality, let this be $(2, 2)$. Select $(2, 2)$. In general, if no two of $(1, 1)$, $(2, 2)$, …, (i, i), $i < 6$ are twins, then in M_i there is a person who is not among their counterparts. Let this person be $(i + 1, i + 1)$ and choose this person. In this way, we can choose (i, i), $i = 1, 2, \ldots, 6$. Denote the counterparts of these 6 persons by X and the first 6 persons of the 7th and 8th rows and columns by A_1, A_2, A_3, A_4, respectively. Let $|X \cap A_i| = a_i$. Then $a_1 + a_2 + a_3 + a_4 \leq 6$, Assume that $a_1 \geq a_2 \geq a_3 \geq a_4$. Suppose $a_1 = 6$. Choose $(7, 7)$ and choose a person in A_2 and a person in A_4 who are not twins and are not the counterpart of $(7, 7)$. These 9 people will cover all the row and columns. If $a_1 < 6$, then $a_2 \leq 3$, $a_3 \leq 2$ and $a_4 \leq 1$. Then there exist $p_i \in A_i$ such that $p_i \notin X$, $i = 1, 2, 3, 4$ and not two of p_1, p_2, p_3, p_4 are twins. Choose these 4 persons as well and the 10 chosen people will cover all the rows and columns.

5. We have

$$(x+y)/3 = (\sqrt{2x-1} + \sqrt{y+3/4} + \sqrt{y+3/4})/3 \le \sqrt{(2x+2y+1/2)/3}.$$

and

$$(x+y) = \sqrt{2x-1} + \sqrt{y+3/4} + \sqrt{y+3/4} \ge \sqrt{2x+2y+1/2}.$$

Let $t = x+y$. Then we have

$$\begin{array}{rcl} 2t^2 - 12t - 3 \le 0 & \Rightarrow & 3 - \sqrt{21/2} \le t \le 3 + \sqrt{21/2} \\ \text{and} \quad 2t^2 - 4t - 1 \ge 0 & \Rightarrow & t \ge 1 + \sqrt{3/2}, \quad t \le 1 - \sqrt{3/2}. \end{array}$$

Since $t \ge 0$, the maximum value is $3 + \sqrt{21/2}$ and the minimum value is $1 + \sqrt{3/2}$. Note that the maximum value is attained by the solution of

$$x + y = 3 + \sqrt{21/2} \quad \text{and} \quad 2x - 1 = \sqrt{y+3/4}$$

while the minimum is attained by the solution of

$$x + y = 1 + \sqrt{3/2} \quad \text{and} \quad y + 3/4 = 0.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 1)

Wednesday, 30 May 2007 **0930-1200**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

1. Let A be any k-element subset of the set $\{1, 2, 3, 4, \cdots, 100\}$. Determine the minimum value of k such that we can always guarantee the existence of two numbers a and b in A such that $|a - b| \leq 4$.

2. Determine the number of those 0-1 binary sequences of ten 0's and ten 1's which do not contain three 0's together.

3. Let A be the set of any 20 points on the circumference of a circle. Joining any two points in A produces one chord of this circle. Suppose every three such chords are not concurrent. Find the number of regions within the circle which are divided by all these chords.

4. In each of the following 7-digit natural numbers:

$$1001011, 5550000, 3838383, 7777777,$$

every digit in the number appears at least 3 times. Find the number of such 7-digit natural numbers.

5. Let $A = \{1, 2, 3, 4, \ldots, 1000\}$. Let m be the number of 2-element subsets $\{a, b\}$ of A such that $a \times b$ is divisible by 6. Find the value of $\lfloor m/10 \rfloor$. (Here and in subsequent questions $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

6. Find the number of non-negative integer solutions of the following inequality:

$$x + y + z + u \leq 20.$$

7. In how many different ways can 7 different prizes be awarded to 5 students in such a way that each student has at least one prize?

8. Let ABC be any triangle. Let D and E be the points respectively in the segments of AB and BC such that $AD = 7DB$ and $BE = 10EC$. Assume that AE and CD meet at point F. Determine $\lfloor k \rfloor$, where k is the real number such that $AF = k \times FE$.

9. Let $S = \{1, 2, 3, 4, \cdots, 50\}$. A 3-element subset $\{a, b, c\}$ of S is said to be *good* if $a + b + c$ is divisible by 3. Determine the number of 3-elements of S which are good.

10. Let $x_1, x_2, \ldots, x_{1970}$ be positive integers satisfying $x_1 + x_2 + \cdots + x_{1970} = 2007$. Determine the largest possible value of $x_1^3 + x_2^3 + \cdots + x_{1970}^3$.

11. Determine the largest value of a such that a satisfies the equations $a^2 - bc - 8a + 7 = 0$ and $b^2 + c^2 + bc - 6a + 6 = 0$ for some real numbers b and c.

12. Determine the number of distinct integers among the numbers

$$\left\lfloor \frac{1^2}{2007} \right\rfloor, \quad \left\lfloor \frac{2^2}{2007} \right\rfloor, \quad \cdots \quad , \quad \left\lfloor \frac{2007^2}{2007} \right\rfloor .$$

13. Determine the number of pairs (a, b) of integers with $1 \leq b < a \leq 200$ such that the sum $(a + b) + (a - b) + ab + a/b$ is a square of a number.

14. This question has been deleted.

15. In an acute-angled triangle ABC, points D, E, and F are the feet of the perpendiculars from A, B, and C onto BC, AC and AB, respectively. Suppose $\sin A = \frac{3}{5}$ and $BC = 39$, find the length of AH, where H is the intersection AD with BE.

16. Let O be the centre of the circumcircle of $\triangle ABC$, P and Q the midpoints of AO and BC, respectively. Suppose $\angle CBA = 4\angle OPQ$ and $\angle ACB = 6\angle OPQ$. Find the size of $\angle OPQ$ in degrees.

17. In $\triangle ABC$, $AC > AB$, the internal angle bisector of $\angle A$ meets BC at D, and E is the foot of the perpendicular from B onto AD. Suppose $AB = 5$, $BE = 4$ and $AE = 3$. Find the value of the expression $\left(\frac{AC+AB}{AC-AB}\right) ED$.

18. Find the value of
$$\prod_{k=1}^{45} \tan(2k-1)^\circ.$$

19. Find the radius of the circle inscribed in a triangle of side lengths $50, 120, 130$.

20. Suppose that $0 < a < b < c < d = 2a$ and
$$(d-a)\left(\frac{a^2}{b-a} + \frac{b^2}{c-b} + \frac{c^2}{d-c}\right) = (a+b+c)^2.$$
Find bcd/a^3.

21. Let f be a function so that
$$f(x) - \frac{1}{2} f\left(\frac{1}{x}\right) = \log x$$
for all $x > 0$, where log denotes logarithm base 10. Find $f(1000)$.

22. Let O be an interior point of $\triangle ABC$. Extend AO to meet the side BC at D. Similarly, extend BO and CO to meet CA and AB respectively at E and F. If $AO = 30$, $FO = 20$, $BO = 60$, $DO = 10$ and $CO = 20$, find EO.

23. For each positive integer n, let a_n denote the number of n-digit integers formed by some or all of the digits 0, 1, 2, and 3 which contain neither a block of 12 nor a block of 21. Evaluate a_9.

24. Let S be any nonempty set of k integers. Find the smallest value of k for which there always exist two distinct integers x and y in S such that $x + y$ or $x - y$ is divisible by 2007.

25. Let P be a 40-sided convex polygon. Find the number of triangles S formed by the vertices of P such that any two vertices of S are separated by at least two other vertices of P.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 1 Solutions)

1. Ans: 21

   If A is the set of multiples of 5 in $\{1, 2, 3, \cdots, 100\}$, then $|A| = 20$ and $|a - b| \geq 5$ for every two numbers in A. Thus, if $|A| = 20$, the existence such two numbers cannot be guaranteed. However, if $|A| = 21$, by the pigeonhole principle, there must be two numbers a, b in one of the following twenty sets:

   $$\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10\}, \cdots, \{96, 97, 98, 99, 100\},$$

   and so $|a - b| \leq 4$. Thus the answer is 21.

2. Ans: 24068

   In such a binary sequence, 0's either appear singly or in blocks of 2. If the sequence has exactly m blocks of double 0's, then there are $10 - 2m$ single 0's. The number of such binary sequences is

   $$\binom{11}{m} \times \binom{11-m}{10-2m}.$$

   Thus the answer is

   $$\sum_{m=0}^{5} \binom{11}{m} \times \binom{11-m}{10-2m} = 24068.$$

3. Ans: 5036

   Consider such a figure as a plane graph G. Then the answer is equal to the number of interior faces of G. The number of vertices in G is $20 + \binom{20}{4}$. The sum of all degrees is

   $$20 \times 21 + 4 \times \binom{20}{4}$$

   and so the number of edges in G is

   $$10 \times 21 + 2 \times \binom{20}{4}.$$

   Hence the number of interior faces, by Euler's formula, is

   $$\binom{20}{4} + 10 \times 21 - 20 + 1 = 5036.$$

Second Solution: Let P_n be the number of such regions with n points on the circumference. Then $P_1 = 1$, $P_2 = 2$ and in general, for $n \geq 2$, $P_{n+1} = P_n + n + \sum_{i=1}^{n-2} i(n-i)$. This can be obtained as follows. Suppose there are $n+1$ points $a_0, \ldots, a_n$ in that order on the circumference. The chords formed by $a_1, \ldots, a_n$ create P_n regions. The chord a_0a_1 adds one region. The chord a_0a_1 adds $1 + 1 \times (n-2)$ regions as this chord intersects the existing chords in $1 \times (n-2)$ points. Similarly, the chord a_0a_2 adds $1 + 2 \times (n-3)$ regions, etc. From this, it is easy to show that

$$\begin{aligned}
P_n &= 1 + \binom{n}{2} + \binom{n-2}{2} + 2\binom{n-3}{2} + \cdots + (n-3)\binom{2}{2} \\
&= 1 + \binom{n}{2} + \binom{n-1}{3} + 1\binom{n-3}{2} + \cdots + (n-4)\binom{2}{2} \\
&= \cdots = 1 + \binom{n}{2} + \binom{n-1}{3} + \binom{n-2}{3} + \cdots + \binom{3}{3} = 1 + \binom{n}{2} + \binom{n}{4}
\end{aligned}$$

Thus $P_{20} = 1 + \binom{20}{2} + \binom{20}{4} = 5036$.

4. Ans: 2844

If only one digit appears, then there are 9 such numbers. If the two digits that appear are both nonzero, then the number of such numbers is

$$2 \times \binom{7}{3}\binom{9}{2} = 2520.$$

If one of two digits that appear is 0, then the number of such numbers is

$$\left(\binom{6}{4} + \binom{6}{3}\right) \times \binom{9}{1} = 315.$$

Hence the answer is $9 + 2520 + 315 = 2844$.

5. Ans: 20791

Let

$$A_6 = \{k \in A : 6 \mid k\}; \quad A_2 = \{k \in A : 2 \mid k, 6 \nmid k\}; \quad A_3 = \{k \in A : 3 \mid k, 6 \nmid k\}.$$

Note that

$$\begin{aligned}
|A_6| &= \left\lfloor \frac{1000}{6} \right\rfloor = 166; \qquad |A_2| = \left\lfloor \frac{1000}{2} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor = 334; \\
|A_3| &= \left\lfloor \frac{1000}{3} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor = 167.
\end{aligned}$$

For the product $a \times b$ to be divisible by 6, either (i) one or both of them are in A_6 or (ii) one is in A_2 and the other is in A_3. Hence

$$m = \binom{166}{2} + 166 \times (1000 - 166) + 334 \times 167 = 207917.$$

6. Ans: 10626
Let $v = 20 - (x + y + z + u)$. Then $v \geq 0$ if and only if $x + y + z + u \leq 20$. Hence the answer is equal to the number of non-negative integer solutions of the following equation:

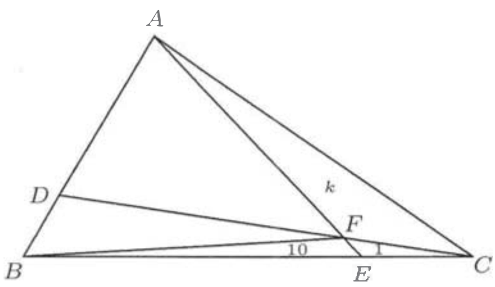
$$x + y + z + u + v = 20$$

and the answer is $\binom{24}{4} = 10626$.

7. Ans: 16800
Either one student receives three prizes; or two students are each awarded two prizes. Thus the answer is

$$\binom{7}{3} \times 5! + \binom{7}{2} \times \binom{5}{2} \times \frac{1}{2} \times 5! = 16800.$$

8. Ans: 77
Assume that $[CEF] = 1$. Then, $[AFC] = k$ and $[BFC] = 11$. Since $AD = 7DB$, $k = [AFC] = 7[BFC] = 77$.



9. Ans: 6544
For $i = 0, 1, 2$, let

$$A_i := \{1 \leq k \leq 50 : 3 \mid (k - i)\}.$$

Then $|A_0| = 16$, $|A_1| = 17$ and $|A_2| = 17$. It can be shown that $3 \mid (a+b+c)$ if and only if either $\{a, b, c\} \subseteq A_i$ for some i or $\{a, b, c\} \cap A_i \neq \emptyset$ for all $i = 1, 2, 3$. Thus the answer is

$$\binom{16}{3} + \binom{17}{3} + \binom{17}{3} \quad 16 \times 17 \times 17 = 6544.$$

10. Ans: 56841
We may assume $x_1 \le x_2 \le \cdots \le x_{1970}$. For $1 \le x_i \le x_j$ with $i \le j$, we have $x_i^3 + x_j^3 \le x_i^3 + x_j^3 + 3(x_j - x_i)(x_j + x_i - 1) = (x_i - 1)^3 + (x_j + 1)^3$. Thus when $x_1 = x_2 = \cdots = x_{1969} = 1$ and $x_{1970} = 38$, the expression $x_1^3 + x_2^3 + \cdots + x_{1970}^3$ attains its maximum value of $1969 + 38^3 = 56841$.

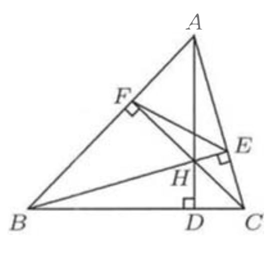
11. Ans: 9
Substituting the first equation $bc = a^2 - 8a + 7$ into the second equation, we have $(b+c)^2 = (a-1)^2$ so that $b + c = \pm(a-1)$. That means b and c are roots of the quadratic equation $x^2 \mp (a-1)x + (a^2 - 8a + 7) = 0$. Thus its discriminant $\triangle = [\mp(a-1)]^2 - 4(a^2 - 8a + 7) \ge 0$, or equivalently, $1 \le a \le 9$. For $b = c = 4$, $a = 9$ satisfies the two equations. Thus the largest value of a is 9.

12. Ans: 1506
Let $a_i = \lfloor i^2/2007 \rfloor$. Note that $\frac{(n+1)^2}{2007} - \frac{n^2}{2007} = \frac{2n+1}{2007} \le 1$ if and only if $n \le 2006/2 = 1003$. Since, $a_1 = 0$, and $a_{1003} = 501$, we see that $a_1, \ldots, a_{1003}$ assume all the values from 0 to 501. We also conclude that $a_{1004}, \ldots, a_{2007}$ are mutually distinct integers. Therefore, the answer is $502 + 1004 = 1506$.

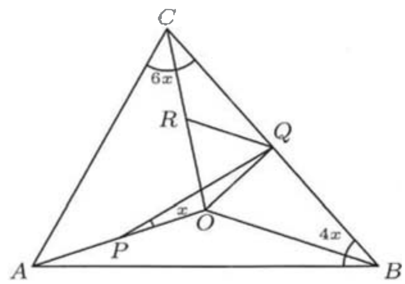
13. Ans: 112
Since the number $(a+b) + (a-b) + ab + a/b = (a/b)(b+1)^2$ is a perfect square with b and $b+1$ relatively prime, the number a/b must be a perfect square. Let $a/b = n^2$. As $a > b$, the number $n \ge 2$ so that $a = bn^2 \le 200$. From this, a can be determined once b and n are chosen. Hence, it suffices to count the number of pairs of (b, n) satisfying $bn^2 \le 200$ with $b \ge 1$ and $n \ge 2$. Hence the answer is $\lfloor 200/2^2 \rfloor + \cdots + \lfloor 200/14^2 \rfloor = 112$.

14. This question has been deleted.

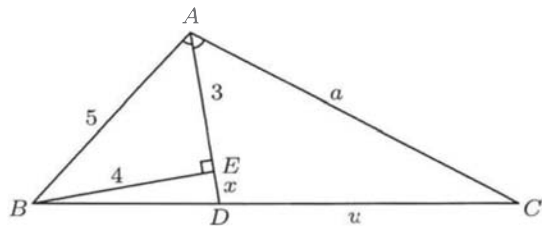
15. Ans: 52
First we know that $AE = AB\cos A$ and $AF = AC\cos A$. By cosine rule, $EF^2 = AE^2 + AF^2 - 2AE \times AF\cos A = \cos^2 A(AB^2 + AC^2 - 2AB \times AC\cos A) = BC^2\cos^2 A$. Therefore $EF = BC\cos A$. It is easy to see that A, E, H, F lie on a circle with diameter AH. Thus $AH = \frac{EF}{\sin A} = BC\cot A = 39 \times \frac{4}{3} = 52$.



16. Ans: 12
Let R be the midpoint of OC. Then $PO = RO = RQ$. Let $\angle OPQ = x$. Then $\angle POR = 2\angle B = 8x$ and $\angle BOA = 2\angle C = 12x$. Also, $\angle ROQ = \angle A = 180° - 10x$. It follows that $\angle PQO = 180° - x - 8x - (180° - 10x) = x$ and therefore $PO = OQ$. Thus $\triangle OQR$ is equilateral, so $\angle ROQ = 180° - 10x = 60°$ and $x = 12°$.



17. Ans: 3
Let $DE = x$, $AC = a > 5$ and $CD = u$. Then $BD = \sqrt{4^2 + x^2}$ and $\cos\angle CAD = \cos\angle BAD = 3/5$. By the cosine rule applied to $\triangle ADC$, we have $u^2 = (3+x)^2 + a^2 - 2a(3+x)(3/5)$. Using the angle bisector theorem, we have $\frac{u^2}{4^2+x^2} = \frac{a^2}{5^2}$. Thus $25[(3+x)^2 + a^2 - 2a(3+x)(3/5)] = a^2(16+x^2)$.



This can be simplified to $(a^2-25)x^2 + 30(a-5)x - 9(a-5)^2 = 0$. Since $a > 5$, we can cancel a common factor $(a-5)$ to get $(a+5)x^2 + 30x - 9(a-5) = 0$, or equivalently $(x+3)((a+5)x - 3(a-5)) = 0$. Thus $x = 3(a-5)/(a+5)$. From this, we obtain $\left(\frac{AC+AB}{AC-AB}\right) ED = \left(\frac{a+5}{a-5}\right) x = 3$.

18. Ans: 1
Note that $\tan(90° - \theta) = 1/\tan\theta$ for $0° < \theta < 90°$and that the product is positive. Setting $j = 46 - k$,

$$\prod_{k=1}^{45} \tan(2k-1)° = \prod_{j=1}^{45} \tan(90 - (2j-1))° = \frac{1}{\prod_{j=1}^{45} \tan(2j-1)°} = 1.$$

19. Ans: 20
The triangle is a right triangle with area $A = 5 \times 120/2 = 300$. The semiperimeter is $s = \frac{1}{2} \times (50 + 120 + 130) = 150$. Hence the inradius is $A/s = 20$.

20. Ans: 4
Set

$$\mathbf{u} = (\sqrt{b-a}, \sqrt{c-b}, \sqrt{d-c})$$
$$\text{and} \quad \mathbf{v} = \Big(\frac{a}{\sqrt{b-a}}, \frac{b}{\sqrt{c-b}}, \frac{c}{\sqrt{d-c}}\Big).$$

Then $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|$ and hence $\mathbf{u} = \alpha \mathbf{v}$ for some $\alpha \geq 0$. Thus

$$\frac{b-a}{a} = \frac{c-b}{b} = \frac{d-c}{c}.$$

Hence $b/a = c/b = d/c$, i.e., a, b, c, d is a geometric progression $a, ra, r^2a, r^3a = 2a$. Now $bcd/a^3 = r^6 = (r^3)^2 = 4$.

21. Ans: 2
Putting $1/x$ in place of x in the given equation yields

$$f\Big(\frac{1}{x}\Big) - \frac{1}{2}f(x) = -\log x.$$

Solving for $f(x)$, we obtain $f(x) = \frac{2}{3}\log x$. Hence $f(1000) = 2$.

22. Ans: 20
We have

$$\frac{OD}{AD} = \frac{[BOC]}{[ABC]}, \quad \frac{OE}{BE} = \frac{[AOC]}{[ABC]}, \quad \frac{OF}{CF} = \frac{[AOB]}{[ABC]}.$$

Thus

$$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1 \quad \Rightarrow \quad OE = 20.$$

23. Ans: 53808

For $n \geq 3$, among the a_n such integers, let b_n denote the number of those that end with 1. By symmetry, the number of those that end with 2 is also equal to b_n. Also the number of those that end with 0 or 3 are both a_{n-1}. Thus

$$a_n = 2a_{n-1} + 2b_n.$$

Among the b_n integers that end with 1, the number of those that end with 11 is b_{n-1} while the number of those that end with 01 or 31 are both a_{n-2}. Thus

$$b_n = b_{n-1} + 2a_{n-2}.$$

Solving, we get $a_n = 3a_{n-1} + 2a_{n-2}$. Since $a_1 = 3$ and $a_2 = 10$, we get $a_9 = 73368$.

24. Ans: 1005

Partition all the possible remainders when divided by 2006 as follows:

$$(0),\ (1, 2006),\ (2, 3005),\ \dots\ ,\ (1003, 1004).$$

Suppose S has 1005 elements. If S has two elements with their difference divisible by 2007, we are done. Otherwise the elements of S have distinct remainders when divided by 2007. By the pigeonhole principle, S has two integers whose sum is divisible by 2007. The set $\{1, 2, \dots, 1004\}$ does not have 2 elements, x, y such that $x + y$ or $x - y$ is divisible by 2007.

25. Ans: 7040

For better understanding, we consider the general case when P has n vertices, where $n \geq 9$. We first count the number of such triangles S having a particular vertex A. The number is $\binom{n-7}{2}$. (This can be obtained as follows. Let the triangle be ABC ordered clockwise. Then B has a "left" neighbour and C has a "right" neighbour. The location of of B and C are uniquely determined by their neighbours. Besides A, B, C and the two vertices to the left and two vertices to the right of A, the two neighbours can be chosen from the remaining $n - 7$ vertices. Thus the required answer is $\frac{n}{3}\binom{n-7}{2} = 7040$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 2)

Saturday, 30 June 2007 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let $a_1, a_2, \ldots, a_n$ be n real numbers whose squares sum to 1. Prove that for any integer $k \geq 2$, there exists n integers $x_1, x_2, \ldots, x_n$, each with absolute value $\leq k-1$ and not all 0, such that
$$\Big|\sum_{i=1}^{n} a_i x_i\Big| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

2. If $a_1,\ a_2,\ \ldots a_n$ are distinct integers, prove that $(x-a_1)(x-a_2)\ldots(x-a_n)-1$ cannot be expressed as a product of two polynomials, each with integer coefficients and of degree at least 1.

3. Let A_1, B_1 be two points on the base AB of an isosceles triangle ABC, with $\angle C > 60^\circ$, such that $\angle A_1CB_1 = \angle ABC$. A circle externally tangent to the circumcircle of $\triangle A_1B_1C$ is tangent to the rays CA and CB at points A_2 and B_2, respectively. Prove that $A_2B_2 = 2AB$.

4. Let $\mathbb{N}$ be the set of positive integers, i.e., $\mathbb{N} = \{1, 2, \ldots\}$. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$f\big(f(m) + f(n)\big) = m + n \quad \text{for all } m, n \in \mathbb{N}.$$

5. Find the largest positive integer x such that x is divisible by all the positive integers $\leq \sqrt[3]{x}$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 2 Solutions)

1. Without loss of generality, we may assume that all the a_i are positive, else we just change the sign of x_i. Since

$$\left(\frac{\sum_{i=1}^n a_i}{n}\right)^2 \leq \frac{\sum_{i=1}^n a_i^2}{n},$$

we have $\sum_{i=1}^n a_i \leq \sqrt{n}$. There are k^n integer sequences $(t_1, t_2, \ldots, t_n)$ satisfying $0 \leq t_i \leq k-1$ and for each such sequence we have $0 \leq \sum_{i=1}^n a_i t_i \leq (k-1)\sqrt{n}$. Now divide the interval $[0, (k-1)\sqrt{n}]$ into $k^n - 1$ equal parts. By the pigeonhole principle, there must exist 2 nonnegative sequences $(y_1, y_2, \ldots, y_n)$ and $(z_1, z_2, \ldots, z_n)$ such that $\left|\sum_{i=1}^n a_i y_i - \sum_{i=1}^n a_i z_i\right| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}$. Set $x_i = y_i - z_i$ to satisfy the condition.

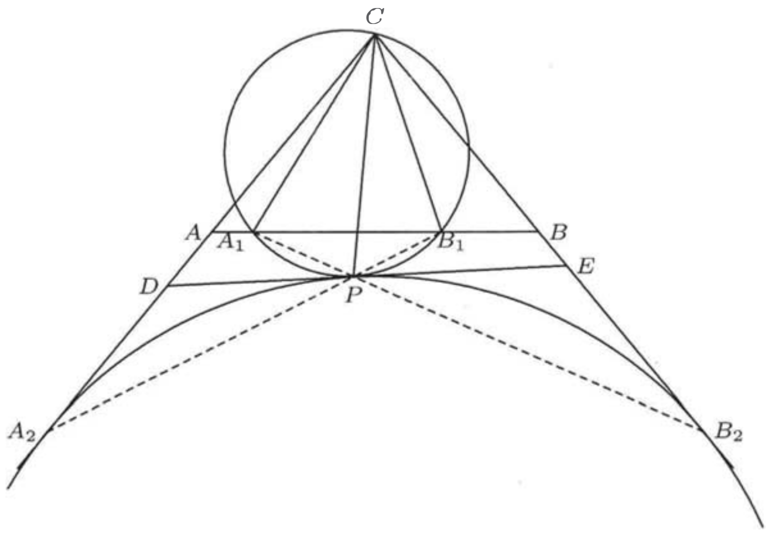
2. Suppose to the contrary that $(x-a_1)(x-a_2)\cdots(x-a_n) - 1 = f(x)g(x)$ for some polynomials $f(x)$ and $g(x)$ with integer coefficients and $\deg(f(x)), \deg(g(x)) \geq 1$. Then $f(a_i)g(a_i) = -1$ for $i = 1, 2, \ldots, n$ implies that $f(a_i) = 1$ and $g(a_i) = -1$ or $f(a_i) = -1$ and $g(a_i) = 1$. Therefore, if we set $h(x) = f(x) + g(x)$, then $h(a_i) = 0$ for all $i = 1, 2, \ldots, n$. As $\deg(h(x)) \leq \max(\deg(f(x)), \deg(g(x))) < n$, the polynomial equation $h(x) = 0$ cannot have n distinct roots. It follows that $h(x)$ must be the zero polynomial. Thus $f(x) = -g(x)$, and therefore

$$(x-a_1)(x-a_2)\ldots(x-a_n) - 1 = -(g(x))^2 \leq 0$$

for all real values of x. But this leads to a contradiction since we can choose a value for x large enough so that $(x-a_1)(x-a_2)\ldots(x-a_n) - 1$ is positive.

2nd solution: We start off as in the first solution. Then instead of defining $h(x)$, we proceed as follows. Let $f(a_i) = 1$, $1 \leq i \leq k$ and $f(a_i) = -1$, $k+1 \leq i \leq n$. Then, $g(a_i) = -1$, $1 \leq i \leq k$ and $g(a_i) = 1$, $k+1 \leq i \leq n$. Therefore $\deg(f(x) - 1) = \deg f(x) \geq \max(k, n-k) \geq \frac{k+(n-k)}{2} = \frac{n}{2}$. Similarly $\deg g(x) \geq \frac{n}{2}$. However $\deg f(x) + \deg g(x) = n$, and thus n is even with $\deg f(x) = \deg g(x) = k = \frac{n}{2}$. Thus $f(x) - 1 = b_1(x-a_1)(x-a_2)\cdots(x-a_k)$, and $g(x) + 1 = b_2(x-a_1)(x-a_2)\cdots(x-a_k)$ for some $b_1, b_2 \in \mathbb{Z}$. Together we get $f(x)g(x) + f(x) - g(x) - 1 = b_1 b_2[(x-a_1)(x-a_2)\cdots(x-a_k)]^2$. By comparing coefficient of the x^n term, $b_1 b_2 = 1$. This give us $f(x) - 1 = g(x) + 1$. Similarly, we have $f(x) + 1 = g(x) - 1$, a contradiction.

3. Let the point of contact of the two circles be P. First we show that A_1, P and B_2 are collinear. Let the common tangent at P meet CA at D and CB at E. Let $\angle ABC = b = \angle CAB = \angle A_1CB_1$, $\angle ACB = c$, $\angle A_1CP = x$ and $\angle B_1CP = y$. Then $x + y = b$ and $2b + c = 180°$. We have $\angle PB_1A = x$, $\angle B_1PE = y$. Therefore, by considering PB_1BE, $\angle BEP = 2x$. Hence $\angle EPB_2 = x$ and consequently, $\angle B_1PB_2 = x + y = b$. This implies that A_1, P and B_2 are collinear. Similarly A_2, P and B_1 are collinear. Then $\triangle A_1BC \sim \triangle A_1CB_1$, and $\triangle CAB_1 \sim \triangle A_1CB_1$, whence $\triangle A_1BC \sim \triangle CAB_1$. Thus $AC/AB_1 = A_1B/BC$ and $BC^2 = A_1B \cdot AB_1$, since $AC = BC$. Also $\triangle AA_2B_1 \sim \triangle BA_1B_2$. Thus $AB_1/AA_2 = BB_2/A_1B$ and whence $AA_2^2 = A_1B \cdot AB_1$. Thus B is the midpoint of CB_2. Since $AB \parallel A_2B_2$, we have $A_2B_2 = 2AB$ as required.



2nd solution: Let Γ_1 be the circumcircle of $\triangle A_1B_1C$ and its centre be O_1, let the other circle Γ_2 has center O_2, and the point of contact of the two circles be P. Now since CA_2 and CB_2 are tangent to Γ_2, we have $CA_2 = CB_2$. Together with $CA = CB$, we have $AA_2 = BB_2$. This implies that $\triangle CAB \sim \triangle CA_2B_2$ and $AB \parallel A_2B_2$. Now $\angle A_2O_2B_2 = 180° - \angle ACB = 2\angle CAB = 2\angle A_1CB_1 = \angle A_1O_1B_1$. Thus, isosceles triangles $A_1O_1B_1$ and $A_2O_2B_2$ are similar. Since $AB \parallel A_2B_2$, $A_1O_1 \parallel O_2B_2$, also note that O_1PO_2 is a straight line. Therefore, we have $2\angle A_1B_1P = \angle A_1O_1P = \angle PO_2B_2 = 2\angle PA_2B_2$. This implies that B_1PA_2 is a straight line. Similarly, A_1PB_2 is a straight line. Now we let Γ_1 intersects CA and CB at D and E respectively, and let DA_1 intersects EB_1 at G. By Pascal's Theorem on Γ_1 and the hexagons CEB_1PA_1D, we have A_2, G and B_2 collinear. Using the fact that $\triangle ACB_1 \sim \triangle CA_1B_1 \sim B_1A_1C$, we have $\angle GA_1B_1 = \angle DA_1A = \angle DCB_1 = \angle CA_1B_1$. Similarly, $\angle GB_1A_1 = \angle CB_1A_1$. This implies that G is the image of C under reflection of line AB. Since G is on A_2B_2, A_2B_2 is twice as far as AB from C. Thus, $A_2B_2 = 2AB$.

3rd solution: Let Γ_1 be the circumcircle of $\triangle A_1B_1C$ and its centre be O_1, let the other circle Γ_2 has center O_2. Now since CA_2 and CB_2 are tangent to Γ_2, we have $CA_2 = CB_2$. This implies that $\triangle CAB \sim \triangle CA_2B_2$. Let us perform inversion with center C and radius CA. Let the image of A_1, A_2, B_1, B_2 under this inversion be A_1', A_2', B_1', B_2' respectively. A, B and C remain invariant. The inversion keeps every line that passes through C invariant. Now the image of the line AA_1B_1B is the circumcircle of $\triangle CAB$, let it be Γ_3, and the image of Γ_1 is the line A_1B_1. Thus the image of Γ_2 is tangent to A_1B_1, AC and BC and is thus the incircle of $\triangle ABC$ and touches the sides AC and BC at A_2' and B_2', respectively. Thus $\frac{CA_2'}{CA} = \frac{CB_2'}{CB} = \frac{1}{2}$, which implies that $\frac{CA_2}{CA} = \frac{CB_2}{CB} = 2$. Hence A and B are the midpoints of A_2C and B_2C, respectively. Thus $A_2B_2 = 2AB$.

4. We show that f is the identity function. First we observe that f is an injective function:

$$\begin{aligned} f(m) = f(n) \quad &\Rightarrow \quad f(m) + f(n) = f(n) + f(n) \\ &\Rightarrow \quad f\big(f(m) + f(n)\big) = f\big(f(n) + f(n)\big) \\ &\Rightarrow \quad m + n = n + n \\ &\Rightarrow \quad m = n \end{aligned}$$

Let $k > 1$ be arbitrary. From the original equation, we have the equations

$$f(f(k+1)+f(k-1)) = (k+1)+(k-1) = 2k, \quad \text{and} \quad f(f(k)+f(k)) = k+k = 2k.$$

Since f is injective, we have

$$f(k+1) + f(k-1) = f(k) + f(k) \quad \text{or} \quad f(k+1) - f(k) = f(k) - f(k-1).$$

This characterizes f as an arithmetic progression, so we may write $f(n) = b + (n-1)t$ where $b = f(1)$ and t is the common difference. The original equation becomes $b + [(b + (m-1)t) + (b + (n-1)t) - 1]t = m + n$, which simplifies to $(3b - 2t - 1) + (m+n)t = m + n$. Comparing coefficients, we conclude that $t = 1$ and $b = 1$. Thus $f(n) = n$, as claimed. Clearly, this function satisfies the original functional equation.

5. The answer is $x = 420$.

Let $p_1, p_2, p_3, \ldots$ be all the primes arranged in increasing order. By Bertrand's Postulate, we have $p_i < p_{i+1} < 2p_i$ for all $i \in \mathbb{N}$, thus we have $p_{k+1} < 2p_k < 4p_{k-1} < 8p_{k-2}$ which implies that $64p_kp_{k-1}p_{k-2} > {p_{k+1}}^3$.

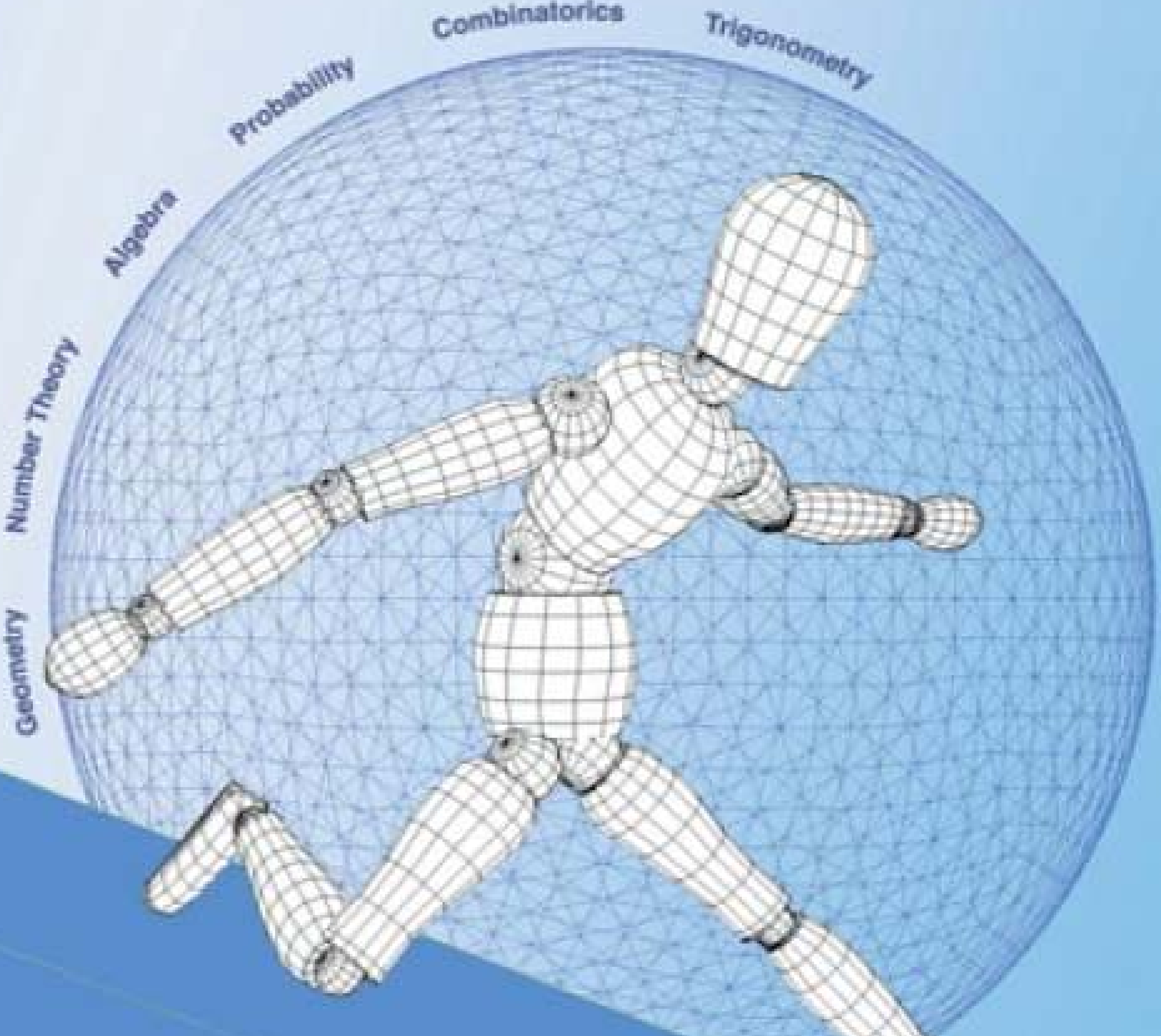
Let $p_k \leq \sqrt[3]{x} < p_{k+1}$ for some $k \in \mathbb{N}$. Note that $p_i \mid x$ for $i = 1, 2, \ldots, k$. Suppose $k \geq 5$, then $\sqrt[3]{x} \geq p_5 = 11$. Since $11 > 2^3$ and $11 > 3^2$, we have $2^33^2 \mid x$. Since $k \geq 5$, $\gcd(p_kp_{k-1}p_{k-2}, 2^33^2) = 1$ and thus $2^33^2p_kp_{k-1}p_{k-2} \mid x$. This means we have $x \geq 72p_kp_{k-1}p_{k-2} > 64p_kp_{k-1}p_{k-2} > {p_{k+1}}^3$, implying $p_{k+1} < \sqrt[3]{x}$, which is a contradiction.Thus $k < 5$ and consequently, $\sqrt[3]{x} < 11$ or $x < 1331$.

Next, we notice that the integer 420 is divisible by all positive integers $\leq \sqrt[3]{420}$, thus $x \geq 420 \Rightarrow \sqrt[3]{x} > 7$. It then follows that x is divisible by $2^2 \cdot 3 \cdot 5 \cdot 7 = 420$.

Finally, suppose $\sqrt[3]{x} \geq 9$. We then have $2^3 \cdot 3^2 \cdot 5 \cdot 7 \mid x$, i.e., $x \geq 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$, which is a contradiction since $x < 1331$. Thus $\sqrt[3]{x} < 9.=$ or $x < 729$. Since $420 \mid x$ and $x < 729$, we have $x = 420$.

Alternatively, since $x < 1331$ and $420 \mid x$, we only need to check the cases $x = 420, 840, 1260$.

SINGAPORE MATHEMATICAL OLYMPIADS
2008



Roger Poh Kheng Siong
Teo Kok Ming
Thomas Teo Teck Kian
Wong Yan Loi

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Singapore Mathematical Olympiad (SMO) 2008

(Junior Section)

Tuesday, 27 May 2008 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answers on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer in the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1 How many zeroes does the product of 25^5, 150^4 and 2008^3 end with?

(A) 5
(B) 9
(C) 10
(D) 12
(E) 13

2 Given that $\sqrt{2x+y} + \sqrt{x^2-9} = 0$, find the value(s) of $y - x$.

(A) -9
(B) -6
(C) -9 or 9
(D) -3 or 3
(E) None of the above

3 The number of integers between 208 and 2008 ending with 1 is

(A) 101
(B) 163
(C) 179
(D) 180
(E) 200

4 The remainder when $7^{2008} + 9^{2008}$ is divided by 64 is

(A) 2
(B) 4
(C) 8
(D) 16
(E) 32

5 John has two 20 cent coins and three 50 cent coins in his pocket. He takes two coins out of his pocket, at random, one after the other without replacement. What is the probability that the total value of the two coins taken out is 70 cents?

(A) $\frac{6}{25}$
(B) $\frac{3}{10}$
(C) $\frac{12}{25}$
(D) $\frac{3}{5}$
(E) $\frac{13}{20}$

6 In the following sum, O represent the digit 0. A, B, X and Y each represents distinct digit. How many possible digits can A be?

$$\begin{array}{r} A\,O\,O\,B\,A\,O\,O\,B \\ +\quad B\,O\,O\,A\,B\,O\,O\,A \\ \hline X\,X\,O\,X\,Y\,X\,O\,X\,X \end{array}$$

(A) 6
(B) 7
(C) 8
(D) 9
(E) 10

7 The least integer that is greater than $(2+\sqrt{3})^2$ is

(A) 13
(B) 14
(C) 15
(D) 16
(E) 17

8 Let x, y and z be non-negative numbers. Suppose $x + y = 10$ and $y + z = 8$. Let $S = x + z$. What is the sum of the maximum and the minimum value of S?

(A) 16
(B) 18
(C) 20
(D) 24
(E) 26

9 How many integer solutions (x, y, z) are there to the equation $xyz = 2008$?

(A) 30
(B) 60
(C) 90
(D) 120
(E) 150

10 The last two digits of 9^{2008} is

(A) 01
(B) 21
(C) 41
(D) 61
(E) 81

Short Questions

11 Find the remainder when $x^{2008} + 2008x + 2008$ is divided by $x + 1$.

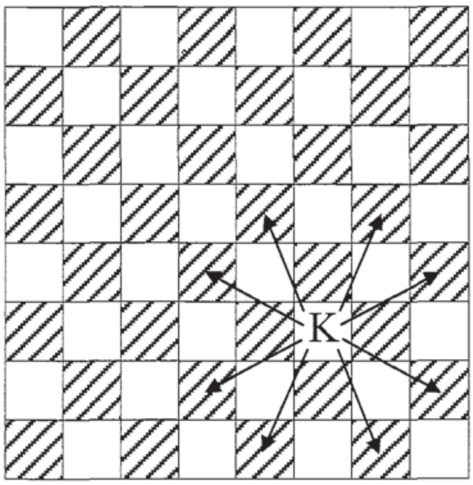
12 Find the maximum value of $\sqrt{x - 144} + \sqrt{722 - x}$.

13 Five identical rectangles of area 8 cm^2 are arranged into a large rectangle as shown.

Find the perimeter of this large rectangle.

14 60 students were interviewed. Of these, 33 liked swimming and 36 liked soccer. Find the greatest possible number of students who neither liked swimming nor soccer.

15 As shown in the picture, the knight can move to any of the indicated squares of the 8×8 chessboard in 1 move. If the knight starts from the position shown, find the number of possible landing positions after 20 consecutive moves.



16 Given that $\alpha + \beta = 17$ and $\alpha\beta = 70$, find the value of $|\alpha - \beta|$.

17 Evaluate (in simplest form)

$$\sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{\ldots}}}}\ .$$

18 Find the sum of all the positive integers less than 999 that are divisible by 15.

19 A brand of orange juice is available in shop A and shop B at an original price of \$2.00 per bottle. Shop A provides the "buy 4 get 1 free" promotion and shop B provides 15% discount if one buys 4 bottles or more. Find the minimum cost (in cents) if one wants to buy 13 bottles of the orange juice.

20 Anna randomly picked five integers from the following list

$$53,\ 62,\ 66,\ 68,\ 71,\ 82,\ 89$$

and discover that the average value of the five integers she picked is still an integer. If two of the integers she picked were 62 and 89, find the sum of the remaining three integers.

21 Suppose the equation $\left| |x - a| - b \right| = 2008$ has 3 distinct real roots and $a \neq 0$. Find the value of b.

22 Find the value of the integer n for the following pair of simultaneous equations to have no solution.

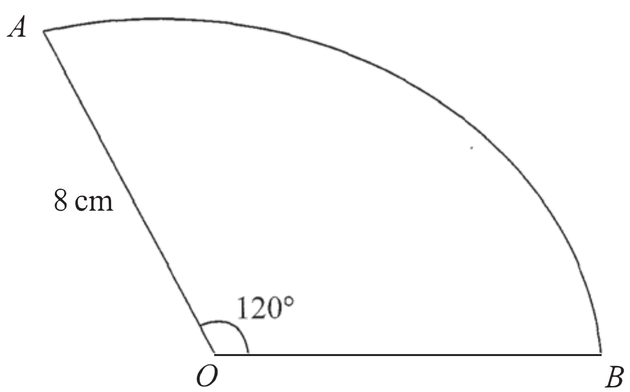
$$\begin{aligned} 2x &= 1 + ny, \\ nx &= 1 + 2y. \end{aligned}$$

23 There are 88 numbers $a_1, a_2, a_3, \ldots, a_{88}$ and each of them is either equals to -3 or -1. Given that $a_1^2 + a_2^2 + \ldots + a_{88}^2 = 280$, find the value of $a_1^4 + a_2^4 + \ldots + a_{88}^4$.

24 Find the value of $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{3}-\frac{1}{4}} \times \dfrac{\frac{1}{4}-\frac{1}{5}}{\frac{1}{5}-\frac{1}{6}} \times \dfrac{\frac{1}{6}-\frac{1}{7}}{\frac{1}{7}-\frac{1}{8}} \times \ldots \times \dfrac{\frac{1}{2004}-\frac{1}{2005}}{\frac{1}{2005}-\frac{1}{2006}} \times \dfrac{\frac{1}{2006}-\frac{1}{2007}}{\frac{1}{2007}-\frac{1}{2008}}$.

25 An integer is chosen from the set $\{1, 2, 3, \ldots, 499, 500\}$. The probability that this integer is divisible by 7 or 11 is $\frac{m}{n}$ in its lowest terms. Find the value of $m + n$.

26 The diagram shows a sector OAB of a circle, centre O and radius 8 cm, in which $\angle AOB = 120°$. Another circle of radius r cm is to be drawn through the points O, A and B. Find the value of r.



27 The difference between the highest common factor and the lowest common multiple of x and 18 is 120. Find the value of x.

28 Let α and β be the roots of $x^2 - 4x + c = 0$, where c is a real number. If $-\alpha$ is a root of $x^2 + 4x - c = 0$, find the value of $\alpha\beta$.

29 Let m, n be integers such that $1 < m \le n$. Define

$$f(m, n) = \left(1 - \frac{1}{m}\right) \times \left(1 - \frac{1}{m+1}\right) \times \left(1 - \frac{1}{m+2}\right) \times \ldots \times \left(1 - \frac{1}{n}\right).$$

If $S = f(2, 2008) + f(3, 2008) + f(4, 2008) + \ldots + f(2008, 2008)$, find the value of $2S$.

30 Let a and b be the roots of $x^2 + 2000x + 1 = 0$ and let c and d be the roots of $x^2 - 2008x + 1 = 0$. Find the value of $(a + c)(b + c)(a - d)(b - d)$.

31 4 black balls, 4 white balls and 2 red balls are arranged in a row. Find the total number of ways this can be done if all the balls of the same colour do not appear in a consecutive block.

32 Given that n is a ten-digit number in the form $\overline{2007x2008y}$ where x and y can be any of the digits 0, 1, 2, ... , 9. How many such numbers n are there that are divisible by 33?

33 In triangle ABC, $AB = (b^2 - 1)$ cm, $BC = a^2$ cm and $AC = 2a$ cm, where a and b are positive integers greater than 1. Find the value of $a - b$.

34 How many positive integers n, where $10 \le n \le 100$, are there such that $\frac{n^2 - 9}{n^2 - 7}$ is a fraction in its lowest terms?

35 Let n be a positive integer such that $n^2 + 19n + 48$ is a perfect square. Find the value of n.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Junior Section Solutions)

1. Answer: (E)
$25^5 = 5^{10}$, $150^4 = 2^4 \times 3^4 \times 5^8$ and $2008^3 = 2^9 \times 251^3$. So the product contains factors 2^{13} and 5^{18}, which produce 10^{13}.

2. Answer: (C)
For $\sqrt{2x+y} + \sqrt{x^2-9} = 0$, $\sqrt{2x+y} = 0$ and $\sqrt{x^2-9} = 0$. So we have $x = 3$ or -3 and $y = -2x = -6$ or $6 \Rightarrow y - x = -9$ or 9.

3. Answer: (D)
The first integer is 211 and the last is 2001. So $211 + (n-1)10 = 2001 \Rightarrow n = 180$.

4. Answer: (A)
$7^{2008} = (8-1)^{2008} = 64k_1 + 1$ for some integers k_1. Similarly, we have $9^{2008} = (8+1)^{2008} = 64k_2 + 1$ for some integers k_2. Hence the remainder is 2.

5. Answer: (D)
$\frac{3}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{3}{4} = \frac{3}{5}$.

6. Answer: (A)
It is clear that $X = 1$. So $A + B = 11 \Rightarrow Y = 2$. Hence $A + B$ can be $3 + 8$ or $4 + 7$ or $5 + 6$ or $6 + 5$ or $7 + 4$ or $8 + 3$.

7. Answer: (B)
$(2+\sqrt{3})^2 = 4 + 2\sqrt{3} + 3 = 7 + 2\sqrt{3}$ and $(2-\sqrt{3})^2 = 7 - 2\sqrt{3}$. So $(2+\sqrt{3})^2 + (2-\sqrt{3})^2 = 14$. Since $0 < 2-\sqrt{3} < 1$, $0 < (2-\sqrt{3})^2 < 1 \Rightarrow$ the least integer that is greater than $(2+\sqrt{3})^2$ is 14.

8. Answer: (C)
$x + y + z = 9 + \frac{S}{2}$. So $x = 1 + \frac{S}{2}$, $y = 9 - \frac{S}{2}$ and $z = -1 + \frac{S}{2}$. Since $x, y, z \geq 0$, we have $2 \leq S \leq 18$.

9. Answer: (D)
$2008 = 2^3 \times 251$. Consider $|x| = 2^{p_1} \times 251^{q_1}$, $|y| = 2^{p_2} \times 251^{q_2}$ and $|z| = 2^{p_3} \times 251^{q_3}$. Then $p_1 + p_2 + p_3 = 3$ and $q_1 + q_2 + q_3 = 1$, so the number of *positive* integer solutions is ${}^5C_2 \times {}^3C_2 = 30$. Together with the 4 possible distributions of the signs:

(+, +, +), (+, −, −), (−, +, −), (−, −, +), the equation has $30 \times 4 = 120$ integer solutions.

10. Answer: (B)
Here we try to find 9^{2008} (mod 100). We see that the 9^2 (mod 100) = 81, 9^4 (mod 100) = 61, 9^6 (mod 100) = 41, 9^8 (mod 100) = 21 and 9^{10} (mod 100) = 01. Hence $9^{2008} \equiv 9^8$ (mod 100) = 21.

11. Answer: 1
The remainder is $(-1)^{2008} + 2008(-1) + 2008 = 1$.

12. Answer: 34
From AM-GM : $\sqrt{ab} \le \dfrac{a+b}{2}$, we have $\sqrt{a} + \sqrt{b} \le \sqrt{2(a+b)}$. Hence $\sqrt{x-144} + \sqrt{722-x} \le \sqrt{2(722-144)} = 34$.

13. Answer: 28
Let width of each rectangle = b, then length = $2b$. So $2b^2 = 8 \Rightarrow b = 2$. Hence perimeter = $14b = 28$.

14. Answer: 24
Let A = {students who liked swimming} and B = {students who liked soccer}. The greatest possible number of students who neither liked swimming nor soccer occurs when $A \subset B$, hence $60 - 36 = 24$.

15. Answer: 32
The knight can move to *any* squares on the chessboard within 20 moves (in fact 4 moves are enough). Note that the knight starts from a white square, so after 20 consecutive moves it can be in any of the $64 \div 2 = 32$ white squares.

16. Answer: 3
Note that α and β are the roots of the equation $x^2 - 17x + 70 = 0$. Solving we have $x = 7$ or 10. Hence $|\alpha - \beta| = 3$.

17. Answer: 2008
Let $x = \sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{\ldots}}}}$, which is clearly positive. Now $x^2 = 2008 + 2007x \Rightarrow (x - 2008)(x + 1) = 0$. Hence the only solution is $x = 2008$.

18. Answer: 33165
The required sum is $15 + 30 + 45 + \ldots + 990 = 15(2211) = 33165$.

19. Ans: 2160
In order to get the best price, the number of bottles bought in shop A should be a multiple of 4. We see that in the 3 cases:

(i)	0 from shop A and 13 from shop B:	$2600 \times 85\% = 2210$,
(ii)	4 from shop A and 8 from shop B:	$800 + 1600 \times 85\% = 2160$,
(iii)	8 from shop A and 3 from shop B:	$1600 + 600 = 2200$,

the lowest price is 2160.

20. Answer: 219
If we take modulo 5, the seven integers 53, **62**, 66, 68, 71, 82, **89** give 3, **2**, 1, 3, 1, 2, **4**. Hence the remaining three integers can only be 66, 71 and 82.

21. Answer: 2008
The equation is equivalent to $|x - a| = b \pm 2008$.

Case 1: If $b < 2008$, then $|x - a| = b - 2008$ has no real root since $b - 2008 < 0$, and $|x - a| = b + 2008$ has at most 2 real roots.

Case 2: If $b > 2008$, then both $|x - a| = b - 2008$ and $|x - a| = b + 2008$ has 2 real roots, which gives 4 *distinct* real roots $a \pm (b - 2008)$ and $a \pm (b + 2008)$, since $a + b + 2008 > a + b - 2008 > a - b + 2008 > a - b - 2008$.

Case 3: If $b = 2008$, then $|x - a| = b - 2008 = 0$ has only 1 real root $x = a$, and $|x - a| = b + 2008 = 4016$ has 2 real roots $x = a \pm 4016$.

Hence $b = 2008$.

22. Answer: -2
Adding the 2 equations and simplifying gives $(n + 2)(x - y) = 2$. Hence if $n = -2$, we get $0 = 2$ which is impossible.

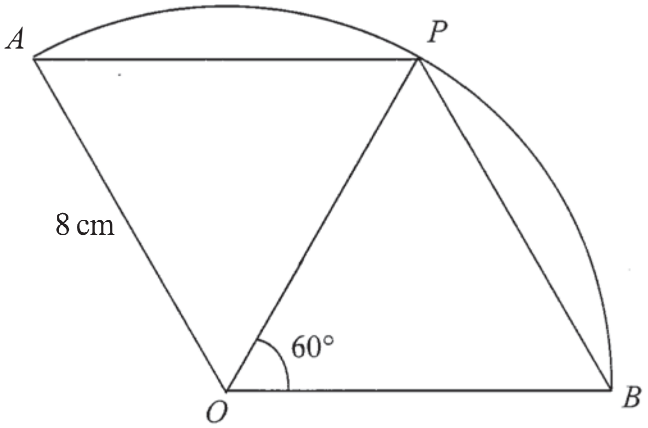
23. Answer: 2008
Let m of them be -3 and n of them be -1. Then $m + n = 88$ and $(-3)^2 m + (-1)^2 n = 280$. Solving, $m = 24$, $n = 64$. Hence $(-3)^4 m + (-1)^4 n = 2008$.

24. Answer: 1004
$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. So the numerator $= \frac{1}{2 \times 3 \times 4 \times 5 \times \ldots \times 2006 \times 2007}$ and the denominator $= \frac{1}{3 \times 4 \times 5 \times 6 \times \ldots \times 2007 \times 2008}$ which gives $\frac{2008}{2} = 1004$.

25. Answer: 61
There are 71 multiples of 7, 45 multiples of 11 and 6 multiples of 77 that are less than 500. So there are $71 + 45 - 6 = 110$ numbers in the set which are multiples of 7 or 11. Hence probability $= \frac{110}{500} = \frac{11}{50}$.

26. Answer: 8

Consider the line OP such that ΔBOP is equilateral. Similarly, ΔAOP is equilateral. Hence $PA = PO = PB \Rightarrow P$ is the centre of the circle that passes through the points O, A and B. Clearly, $r = OP = 8$ cm.



27. Answer: 42

Let the HCF of x & 18 be k and let $x = ka$, $18 = kb$ where gcd $(a, b) = 1$. We have LCM of x & 18 = kab. From information given, $kab - k = 120$. Clearly gcd $(ab - 1, b) = 1$, so gcd $(kab - k, kb) = k \Rightarrow$ gcd $(120, 18) = k \Rightarrow k = 6$. Hence $b = 3$ and $x = 42$.

28. Answer: 0

α is a root of the equation $x^2 - 4x + c = 0$ so $\alpha^2 - 4\alpha + c = 0$ and $-\alpha$ is a root of the equation $x^2 + 4x - c = 0$ so $(-\alpha)^2 + 4(-\alpha) - c = 0$. We have $2c = 0 \Rightarrow \alpha\beta = c = 0$.

29. Answer: 2007

$$f(m, n) = \left(\frac{m-1}{m}\right) \times \left(\frac{m}{m+1}\right) \times \left(\frac{m+1}{m+2}\right) \times \ldots \times \left(\frac{n-1}{n}\right) = \frac{m-1}{n}.$$

$$\Rightarrow f(2, n) + f(3, n) + f(4, n) + \ldots + f(n, n) = \frac{1 + 2 + 3 + \ldots + (n-1)}{n} = \frac{n-1}{2}.$$

$$\Rightarrow f(2, 2008) + f(3, 2008) + f(4, 2008) + \ldots + f(2008, 2008) = \frac{2007}{2} = S.$$

30. Answer: 32064

Note that $ab = cd = 1$, $a + b = -2000$ and $c + d = 2008$. So we have

$$\begin{aligned}(a+c)(b+c)(a-d)(b-d) &= [ab + (a+b)c + c^2][ab - (a+b)d + d^2] \\ &= (1 - 2000c + c^2)(1 + 2000d + d^2) \\ &= (8c)(4008d) \\ &= 32064.\end{aligned}$$

31. Answer: 2376

Using the Principle of Inclusion and Exclusion, number of ways $= \frac{10!}{4!4!2!} - 2\frac{7!}{4!2!} - \frac{9!}{4!4!} + 2\frac{6!}{4!} + \frac{4!}{2!} - 3! = 3150 - 210 - 630 + 60 + 12 - 6 = 2376$.

32. Answer: 3

$33 = 3 \times 11$. So 3 must divide $2 + 7 + x + 2 + 8 + y \Rightarrow 19 + x + y = 3k_1$. 11 must divide $(7 + 2 + y) - (2 + x + 8) \Rightarrow y - x - 1 = 11k_2 \Rightarrow -10 \le 11k_2 \le 8 \Rightarrow k_2 = 0$. So $y = x + 1 \Rightarrow 20 + 2x = 3k_1$. When $k_1 = 8$, $x = 2$ and $y = 3$. When $k_1 = 10$, $x = 5$ and $y = 6$. When $k_1 = 12$, $x = 8$ and $y = 9$.

33. Answer: 0

Using Triangle Inequality, $a^2 + 2a > b^2 - 1 \Rightarrow (a + 1 - b)(a + 1 + b) > 0$. Since $a + 1 + b > 0 \Rightarrow a + 1 - b > 0 \Rightarrow a - b \ge 0$. Using Triangle Inequality again, $2a + b^2 - 1 > a^2 \Rightarrow (a - 1 - b)(a - 1 + b) < 0$. Since $a - 1 + b > 0 \Rightarrow a - 1 - b < 0 \Rightarrow a - b \le 0$. Hence $a - b = 0$.

34. Answer: 46

$\gcd(n^2 - 9, n^2 - 7) = 1 \Rightarrow \gcd(n^2 - 9, 2) = 1$. Hence $n^2 - 9$ must be an odd number $\Rightarrow n$ is even. Since $10 \le n \le 100$, there are $\frac{100 - 10}{2} + 1 = 46$ possible positive integers n.

35. Answer: 33

$n^2 + 19n + 48 = (n + 3)(n + 16)$. If $n + 3 = 13k$ for some integer k, then $n + 16 = 13(k + 1)$. However, $(n + 3)(n + 16) = 169k(k + 1)$ cannot be a square. So 13 must not divide $n + 3 \Rightarrow \gcd(n + 3, n + 16) = 1 \Rightarrow$ they should be both perfect squares. Let $n + 3 = m^2$. Since $n + 16 \ge (m + 1)^2 = m^2 + 2m + 1 = n + 3 + 2m + 1$, we have $m \le 6$. Checking $m = 2, 3, 4, 5$ and 6, $n + 16 = m^2 + 13 = 17, 22, 29, 38$ and 49 respectively. Hence $m = 6$ and $n = 33$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Junior Section, Round 2)

Saturday, 28 June 2008 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. In $\triangle ABC$, $\angle ACB = 90^\circ$, D is the foot of the altitude from C to AB and E is the point on the side BC such that $CE = BD/2$. Prove that $AD + CE = AE$.

2. Let a, b, c, d be positive real numbers such that $cd = 1$. Prove that there is an integer n such that $ab \leq n^2 \leq (a+c)(b+d)$.

3. In the quadrilateral $PQRS$, A, B, C and D are the midpoints of the sides PQ, QR, RS and SP respectively, and M is the midpoint of CD. Suppose H is the point on the line AM such that $HC = BC$. Prove that $\angle BHM = 90^\circ$.

4. Six distinct positive integers a, b, c, d, e, f are given. Jack and Jill calculated the sums of each pair of these numbers. Jack claims that he has 10 prime numbers while Jill claims that she has 9 prime numbers among the sums. Who has the correct claim?

5. Determine all primes p such that
$$5^p + 4 \cdot p^4$$
is a perfect square, i.e., the square of an integer.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Junior Section, Round 2 Solutions)

1. Let $AD = q$, $BD = p$. Then $CE = p/2$ and $p + q = AB = c$. By similar triangles, we also have $CA = b = \sqrt{cq}$. Let F be the point on AE so that $EF = CE$. From $\triangle ACE$, we have

$$AE^2 = CA^2 + CE^2, \quad \text{i.e.,} \quad (AF + p/2)^2 = cq + p^2/4 = (p + q)q + p^2/4.$$

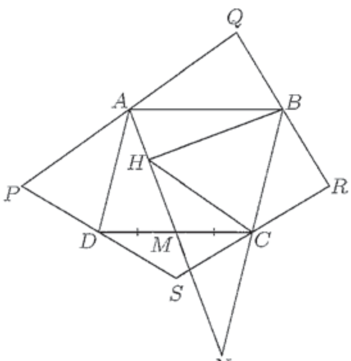
Therefore

$$AF(AF + p) = q(q + p).$$

From here it follows that $AF = q$ and we are done.

2. We need to prove that $\sqrt{ab} + 1 \leq \sqrt{(a+c)(b+d)}$. By squaring both sides and simplify, this is equivalent to $\sqrt{ab} \leq (bc + ad)/2$. Since $(bc + ad)/2 \geq \sqrt{abcd} = \sqrt{ab}$, the proof is complete.

3. First observe that $ABCD$ is a parallelogram by Varignon's theorem. Let the extensions of AM and BC meet at N. Since AD is parallel to CN, $\angle MAD = \angle MNC$. Since $\angle AMD = \angle NMC$ and $MD = MC$, $\triangle MAD$ and $\triangle MNC$ are congruent, so that $CN = DA = CB = HC$. Thus H lies on the circle centred at C with diameter BN. Hence $\angle BHM = 90°$.



4. Suppose k of the 6 numbers are even. Since the sum of two even or two odd numbers are even, and the sum of two distinct positive positive integers is > 2, the only even prime, we see that the number of primes among the sums is at most $k(6-k)$. By checking for $k = 0, 1, \ldots, 6$, we see that the maximum value of $k(6-k)$ is 9 attained when $k = 3$. Thus Jack's answer is definitely wrong. Jill's answer is correct because 9 primes can be obtained from the following 6 numbers: 2, 4, 8, 3, 15, 39.

5. Let $5^p + 4 \cdot p^4 = q^2$. Then

$$5^p = (q - 2p^2)(q + 2p^2).$$

Thus

$$q - 2p^2 = 5^s, \quad q + 2p^2 = 5^t \quad \text{where } 0 \le s < t \text{ and } s + t = p$$

Eliminating q, we get $4p^2 = 5^s(5^{t-s} - 1)$. If $s > 0$, then $5 \mid 4p^2$. Thus $p = 5$ and the given expression is indeed a square. If $s = 0$, then $t = p$ and we have $5^p = 4p^2 + 1$. We shall prove by induction that $5^k > 4k^2 + 1$ for every integer $k \ge 2$. The inequality certainly holds for $k = 2$. So we assume that it holds for some $k \ge 2$. Note that

$$\frac{4(k+1)^2 + 1}{4k^2 + 1} = \frac{4k^2 + 1}{4k^2 + 1} + \frac{8k}{4k^2 + 1} + \frac{4}{4k^2 + 1} < 1 + 1 + 1 < 5 \quad \text{for } k \ge 2.$$

Therefore

$$5^{k+1} = 5 \cdot 5^k > 5(4k^2 + 1) > 4(k+1)^2 + 1.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Senior Section)

Tuesday, 27 May 2008 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D *or* E) *corresponding to the correct answer.*

For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Find the value of $\frac{1+3+5+7+...+99}{2+4+6+8+...+100}$.

(A) $\frac{48}{49}$

(B) $\frac{49}{50}$

(C) $\frac{50}{51}$

(D) $\frac{98}{99}$

(E) $\frac{100}{101}$

2. Suppose that x and y are real numbers that satisfy all the following three conditions: $3x-2y=4-p$; $4x-3y=2+p$; $x>y$. What are the possible values of p?

(A) $p>-1$

(B) $p<1$

(C) $p<-1$

(D) $p>1$

(E) p can be any real number

3. If $f(x)=x^2+\sqrt{1-x^2}$ where $-1\le x\le 1$, find the range of $f(x)$.

(A) $\frac{1}{2}\le f(x)\le 1$

(B) $1\le f(x)\le \frac{5}{4}$

(C) $1\le f(x)\le \frac{1+2\sqrt{3}}{4}$

(D) $\frac{\sqrt{3}}{2}\le f(x)\le 1$

(E) $\frac{1}{2}\le f(x)\le \frac{\sqrt{3}}{2}$

4. If a and b are integers and $\sqrt{7-4\sqrt{3}}$ is one of the roots of the equation $x^2 + ax + b = 0$, find the value of $a + b$.

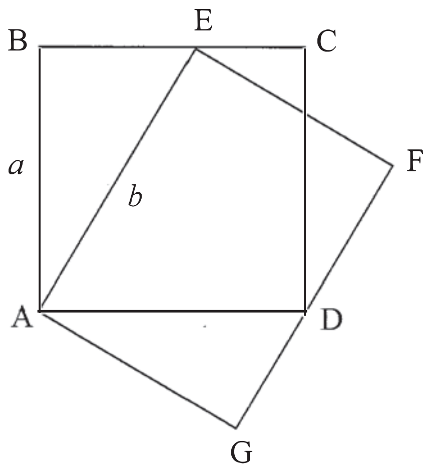
(A) -3
(B) -2
(C) 0
(D) 2
(E) 3

5. A bag contains 30 balls that are numbered 1, 2, …, 30. Two balls are randomly chosen from the bag. Find the probability that the sum of the two numbers is divisible by 3.

(A) $\frac{1}{2}$
(B) $\frac{1}{3}$
(C) $\frac{7}{29}$
(D) $\frac{9}{29}$
(E) $\frac{11}{87}$

6. ABCD is a square with AB = a, and AEFG is a rectangle such that E lies on side BC and D lies on side FG. If AE = b, what is the length of side EF?

(A) $\frac{b}{a}$
(B) $\frac{3a^2}{2b}$
(C) $\frac{4a^2}{3b}$
(D) $\frac{\sqrt{2}\,a^2}{b}$
(E) $\frac{a^2}{b}$

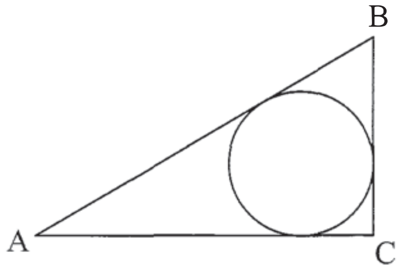


7. Find the value of $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$.

(A) 1
(B) $\frac{3}{2}$
(C) $\frac{7}{4}$
(D) 2
(E) $\frac{5}{2}$

8. A circle with radius x cm is inscribed inside a triangle ABC, where $\angle$ ACB is a right angle. If AB = 9 cm and the area of the triangle ABC is 36 cm^2, find the value of x.

(A) 2.2
(B) 2.6
(C) 3
(D) 3.4
(E) 3.8



9. How many positive integers n are there such that $7n + 1$ is a perfect square and $3n + 1 < 2008$?

(A) 6
(B) 9
(C) 12
(D) 15
(E) 18

10. Find the minimum value of $(a+b)\left(\frac{1}{a}+\frac{4}{b}\right)$, where a and b range over all positive real numbers.

(A) 3
(B) 6
(C) 9
(D) 12
(E) 15

Short Questions

11. Find the smallest integer n such that $n(\sqrt{101}-10)>1$.

12. Given that x and y are positive real numbers such that $(x+y)^2 = 2500$ and $xy = 500$, find the exact value of $x^3 + y^3$.

13. Find the smallest positive integer N such that $2^n > n^2$ for every integer n in $\{N, N+1, N+2, N+3, N+4\}$.

14. The lengths of the sides of a quadrilateral are 2006 cm, 2007 cm, 2008 cm and x cm. If x is an integer, find the largest possible value of x.

15. Find the number of positive integers x that satisfy the inequality $\left|3+\log_x \frac{1}{3}\right| < \frac{8}{3}$.

16. Two bullets are placed in two consecutive chambers of a 6-chamber pistol. The cylinder is then spun. The pistol is fired but the first shot is a blank. Let p denote the probability that the second shot is also a blank if the cylinder is spun after the first shot and let q denote the probability that the second shot is also a blank if the cylinder is not spun after the first shot. Find the smallest integer N such that $N \geq \frac{100p}{q}$.

17. Find the value of $\left(\log_{\sqrt{2}}(\cos 20^\circ)+\log_{\sqrt{2}}(\cos 40^\circ)+\log_{\sqrt{2}}(\cos 80^\circ)\right)^2$.

18. Find the number of ways for 5 persons to be seated at a rectangular table with 6 seats, 2 each on the longer sides and 1 each on the shorter sides. The seats are not numbered.

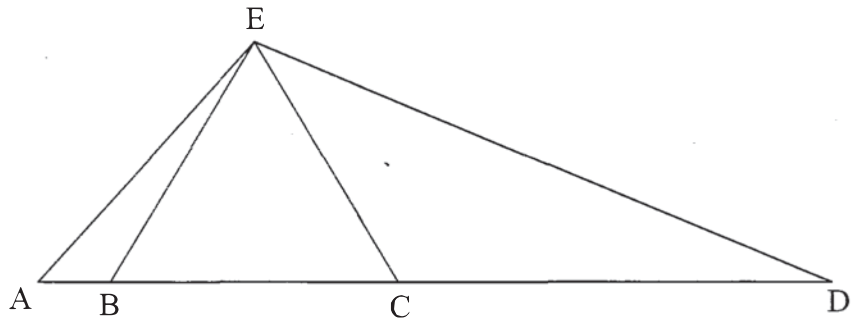
19. Find the remainder when $(x-1)^{100}+(x-2)^{200}$ is divided by x^2-3x+2.

20. Suppose that ABC is a triangle and D is a point on side AB with AD = BD = CD. If $\angle ACB = x^\circ$, find the value of x.

21. If x, y and z are positive integers such that $27x + 28y + 29z = 363$, find the value of $10(x + y + z)$.

22. Find the value of $\dfrac{\tan 40^\circ \tan 60^\circ \tan 80^\circ}{\tan 40^\circ + \tan 60^\circ + \tan 80^\circ}$.

23. In the figure below, ADE is a triangle with $\angle$AED $=120^\circ$, and B and C are points on side AD such that BCE is an equilateral triangle. If AB = 4 cm, CD = 16 cm and BC $= x$ cm, find the value of x.



24. Suppose that x and y are positive integers such that $x + 2y = 2008$ and xy has the maximum value. Find the value of $x - y$.

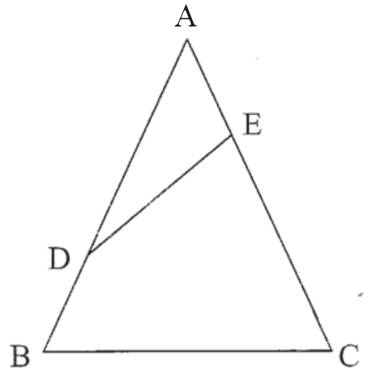
25. If $\cos(2A) = -\dfrac{\sqrt{5}}{3}$, find the value of $6\sin^6 A + 6\cos^6 A$.

26. Let N be the positive integer for which the sum of its two smallest factors is 4 and the sum of its two largest factors is 204. Find the value of N.

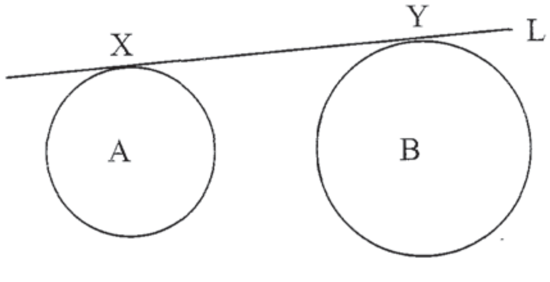
27. If $S = \sum_{k=1}^{99} \dfrac{(-1)^{k+1}}{\sqrt{k(k+1)}\,(\sqrt{k+1} - \sqrt{k})}$, find the value of $1000S$.

28. A teacher wrote down three positive integers on the whiteboard: 1125, 2925, N, and asked her class to compute the least common multiple of the three numbers. One student misread 1125 as 1725 and computed the least common multiple of 1725, 2925 and N instead. The answer he obtained was the same as the correct answer. Find the least possible value of N.

29. The figure below shows a triangle ABC where AB = AC. D and E are points on sides AB and AC, respectively, such that AB = 4DB and AC = 4AE. If the area of the quadrilateral BCED is 52 cm^2 and the area of the triangle ADE is x cm^2, find x.



30. The figure below shows two circles with centres A and B, and a line L which is a tangent to the circles at X and Y. Suppose that XY = 40 cm, AB = 41 cm and the area of the quadrilateral ABYX is 300 cm^2. If a and b denote the areas of the circles with centre A and centre B respectively, find the value of $\frac{b}{a}$.



31. Find the maximum value of $3\sin\left(x+\frac{\pi}{9}\right)+5\sin\left(x+\frac{4\pi}{9}\right)$, where x ranges over all real numbers.

32. Find the number of 11-digit positive integers such that the digits from left to right are non-decreasing. (For example, 12345678999, 55555555555, 23345557889.)

33. Find the largest positive integer n such that $\sqrt{n-100}+\sqrt{n+100}$ is a rational number.

34. Let $S = \{ 1, 2, 3, \ldots, 20 \}$ be the set of all positive integers from 1 to 20. Suppose that N is the smallest positive integer such that exactly eighteen numbers from S are factors of N, and the only two numbers from S that are not factors of N are consecutive integers. Find the sum of the digits of N.

35. Let $a_1, a_2, a_3, \ldots$ be the sequence of all positive integers that are relatively prime to 75, where $a_1 < a_2 < a_3 < \cdots$. (The first five terms of the sequence are: $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 7$, $a_5 = 8$.) Find the value of a_{2008}.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Senior Section Solutions)

1. Answer: (C)
Since the numbers of terms in the numerator and denominator are the same, we have $\dfrac{1+3+5+7+\cdots+99}{2+4+6+8+\cdots+100}=\dfrac{100}{102}=\dfrac{50}{51}$.

2. Answer: (D)
Solving the simultaneous equations $3x-2y=4-p$ and $4x-3y=2+p$, we obtain $x=8-5p$, $y=10-7p$. Since $x>y$, $8-5p>10-7p$, so $p>1$.

3. Answer: (B)
Let $x^2+\sqrt{1-x^2}=M$. Then $(x^2-M)^2=1-x^2$, which leads to $x^4-(2M-1)x^2+M^2-1=0$. For real x, we must have $(2M-1)^2-4(M^2-1)>0$, which gives $M\le\dfrac{5}{4}$.
Note that $f(-1)=f(0)=f(1)=1$. Assume that $f(x)<1$ for some $x\ne -1,0,1$. Then $x^2+\sqrt{1-x^2}<1$, or $\sqrt{1-x^2}<1-x^2$, which is not possible since $-1<x<1$. Thus $f(x)\ge 1$, and hence $1\le f(x)\le\dfrac{5}{4}$.

4. Answer: (A)
Note that $\sqrt{7-4\sqrt{3}}=\sqrt{(2-\sqrt{3})^2}=2-\sqrt{3}$. Since $\sqrt{7-4\sqrt{3}}$ is a root of the equation $x^2+ax+b=0$, we have $(7-4\sqrt{3})+(2-\sqrt{3})a+b=0$. Rearranging the terms, we obtain $(7+2a+b)-(4+a)\sqrt{3}=0$. This implies that $7+2a+b=0$ and $4+a=0$, which give $a=-4$ and $b=1$. Thus $a+b=-3$.

5. Answer: (B)
Let $S_1=\{1,4,7,\ldots,28\}$, $S_2=\{2,5,8,\ldots,29\}$, $S_3=\{3,6,9,\ldots,30\}$. Note that each of the three sets has 10 elements. Now for any two distinct numbers a and b in $\{1,2,3,\ldots,28,29,30\}$, their sum $a+b$ is divisible by 3 if and only if one of the following conditions holds:
   (i) the two numbers a and b belongs to S_1 and S_2 respectively;
   (ii) both numbers a and b belong to S_3.

Therefore the required probability $=\dfrac{{}^{10}C_2+{}^{10}C_1\,{}^{10}C_1}{{}^{10}C_3}=\dfrac{45+100}{(15)(29)}=\dfrac{1}{3}$.

6. Answer: (E)

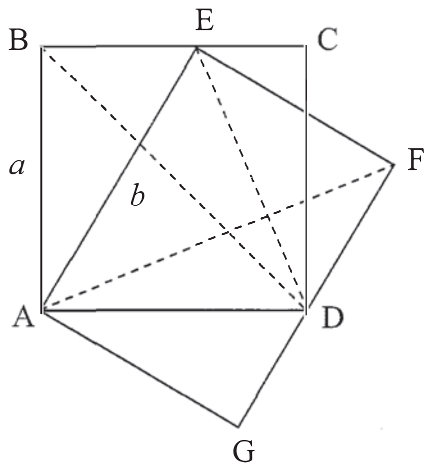
Note that

area of ΔABD = area of ΔAED
= area of ΔAEF.

Therefore

a^2 = area of square ABCD
= 2 × area of ΔABD
= 2 × area of ΔAEF
= area of rectangle AEFG.

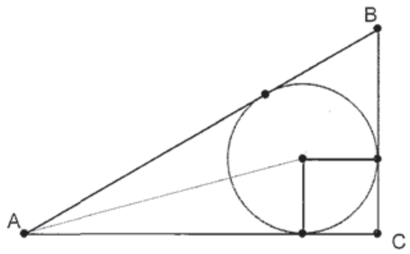
Hence $\mathrm{EF} = \dfrac{a^2}{b}$.



7. Answer: (B)

$$\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$$
$$=2\left(\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}\right)=2\left(\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}\right)$$
$$=2\left[\left(\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right)^2-2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right]$$
$$=2\left(1-\frac{1}{2}\sin^2\frac{\pi}{4}\right)=\frac{3}{2}.$$

8. Answer: (C)



Note that area of ΔABC = (AB)$x + x^2$. Thus we obtain $36 = 9x + x^2$, or $x^2+9x-36=0$. Solving the equation gives $x = 3$ or $x = -12$, so the radius is 3cm.

9. Answer: (E)

First note that $3n + 1 < 2008$ implies that $n < 669$.

Since $7n + 1$ is a perfect square, we let $7n + 1 = m^2$. Then $n=\dfrac{m^2-1}{7}$, which is an integer. Thus 7 is a factor of $m^2-1=(m-1)(m+1)$. Since 7 is a prime number, this implies that 7 is a factor of $m-1$ or $m+1$; that is, $m = 7k-1$ or $7k+1$ for some integer k.

If $m = 7k-1$, then $n = \dfrac{m^2-1}{7} = \dfrac{49k^2-14k}{7} = 7k^2-2k$. As $n < 669$, this gives $7k^2-2k < 669$, which implies that $k = 1, 2, \ldots, 9$.

Similarly, if $m = 7k+1$, then $n = \dfrac{m^2-1}{7} = \dfrac{49k^2+14k}{7} = 7k^2+2k < 669$, so we obtain $k = 1, 2, 3, \ldots, 9$.

Hence there are 18 such positive integers.

10. Answer: (C)

$(a+b)\left(\dfrac{1}{a}+\dfrac{4}{b}\right) = 1+4+\dfrac{b}{a}+\dfrac{4a}{b} = 5+\dfrac{b^2+4a^2}{ab}$. Let $m = \dfrac{b^2+4a^2}{ab}$. Then we have $4a^2 - mba + b^2 = 0$. This implies that for any positive values of m and b, $m^2b^2 - 16b^2 \geq 0$. Thus we have $m^2 - 16 \geq 0$, or $m \geq 4$.

Therefore $(a+b)\left(\dfrac{1}{a}+\dfrac{4}{b}\right) = 5+m \geq 9$. Hence the minimum value is 9.

11. Answer: 21

Solving the inequality $n(\sqrt{101}-10) > 1$, we obtain

$$n > \frac{1}{\sqrt{101}-10} = \frac{\sqrt{101}+10}{1} = \sqrt{101}+10.$$

As n is an integer and $10 < \sqrt{101} < 11$, we have $n \geq 21$.

12. Answer: 50000

$$\begin{aligned} x^3+y^3 &= (x+y)(x^2-xy+y^2) = (x+y)[(x+y)^2-3xy] \\ &= 50[2500-3(500)] = 50000. \end{aligned}$$

13. Answer: 5

Using guess and check, we obtain $N = 5$.

14. Answer: 6020

By triangle inequality, $x < 2006 + 2007 + 2008 = 6021$. Since x is an integer, it follows that the largest possible value of x is 6020.

15. Answer: 25

We have $\quad -\dfrac{8}{3} < 3 + \log_x \dfrac{1}{3} < \dfrac{8}{3}$

$$-\frac{17}{3} < -\frac{1}{\log_3 x} < -\frac{1}{3}$$
$$\frac{3}{17} < \log_3 x < 3$$
$$3^{3/17} < x < 27.$$
As $1 < 3^{3/17} < 2$, we see that integer solutions of the last inequality are $x = 2$, 3, … , 26. Hence there are 25 of them.

16. Answer: 89
p = Prob(2nd shot blank | cylinder spun again) = 4/6, and
q = Prob(2nd shot blank | cylinder not spun again) = 3/4.
Therefore $\frac{100p}{q} = \frac{800}{9} \approx 88.8$, and hence the smallest $N = 89$.

17. Answer: 36
We have
$$\log_{\sqrt{2}}(\cos 20^\circ) + \log_{\sqrt{2}}(\cos 40^\circ) + \log_{\sqrt{2}}(\cos 80^\circ)$$
$$= \log_{\sqrt{2}}(\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ)$$
$$= \log_{\sqrt{2}}\left(\cos 20^\circ \cdot \frac{1}{2}(\cos 120^\circ + \cos 40^\circ)\right)$$
$$= \log_{\sqrt{2}}\left(-\frac{1}{4}\cos 20^\circ + \frac{1}{2}\cos 40^\circ \cos 20^\circ\right)$$
$$= \log_{\sqrt{2}}\left(-\frac{1}{4}\cos 20^\circ + \frac{1}{4}(\cos 60^\circ + \cos 20^\circ)\right)$$
$$= \log_{\sqrt{2}}\frac{1}{8} = -\frac{\log_2 8}{\log_2(2^{1/2})} = -6.$$
Hence the answer is $(-6)^2 = 36$.

18. Answer: 360
The empty seat can be considered to be a person. Now the number of ways of sitting at a round table is (6 – 1)! = 120. Note that each arrangement at a round table can be matched to 3 different arrangements at the rectangular table. Thus the number of ways of sitting at the rectangular table is 3 × 120 = 360.

19. Answer: 1
By the division algorithm, we have
$$(x-1)^{100} + (x-2)^{200} = (x^2 - 3x + 2)q(x) + ax + b,$$
where $q(x)$ is the quotient, and $ax + b$ is the remainder with constants a and b.
Note that $x^2 - 3x + 2 = (x-1)(x-2)$. Therefore
$$(x-1)^{100} + (x-2)^{200} = (x-1)(x-2)q(x) + ax + b,$$

Putting $x = 1$, we obtain

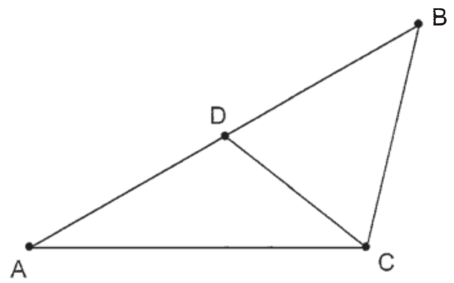
$$(1-2)^{200} = a+b, \text{ or } a+b = 1.$$

Putting $x = 2$, we obtain

$$(2-1)^{100} = 2a+b, \text{ or } 2a+b = 1.$$

Solving the two equations gives $a = 0$ and $b = 1$.

20. Answer: 90



First note that since ΔDAC and ΔDBC are isosceles triangles, we have $\angle$DAC = $\angle$DCA and $\angle$DBC = $\angle$DCB. Now by considering the sum of angles in ΔABC, we have

$$\angle DAC + \angle DBC + \angle DCA + \angle DCB = 180^\circ,$$
$$2\angle DCA + 2\angle DCB = 180^\circ.$$

Therefore $\angle ACB = \angle DCA + \angle DCB = 90^\circ$.

21. Answer: 130

Since x, y and z are positive, we have

$$27(x+y+z) < 27x + 28y + 29z < 29(x+y+z).$$

That is, $27(x+y+z) < 363 < 29(x+y+z)$.

Therefore it follows that

$$x+y+z < 13.4 \text{ and } x+y+z > 12.5.$$

Since x, y and z are integers, we obtain $x+y+z = 13$. Hence $10(x+y+z) = 130$.

22. Answer: 1

We show that more generally, if ΔABC is acute-angled, then

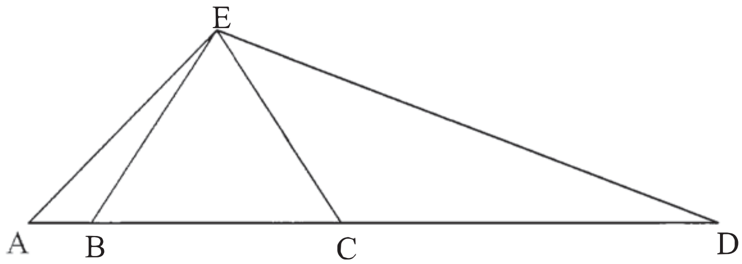
$$\frac{\tan A \cdot \tan B \cdot \tan C}{\tan A + \tan B + \tan C} = 1.$$

We have

$$\begin{aligned}
&\tan A + \tan B + \tan C \\
&= \tan A + \tan B + \tan(180^\circ - (A+B)) \\
&= \tan A + \tan B - \tan(A+B) \\
&= \tan A + \tan B - \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
&= (\tan A + \tan B)\left(1 - \frac{1}{1 - \tan A \cdot \tan B}\right)
\end{aligned}$$

$$= (\tan A + \tan B)\left(\frac{-\tan A \cdot \tan B}{1 - \tan A \cdot \tan B}\right)$$
$$= (\tan A \cdot \tan B)\left(-\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}\right)$$
$$= \tan A \cdot \tan B \cdot \tan(180^\circ - (A + B))$$
$$= \tan A \cdot \tan B \cdot \tan C,$$
so the result follows.

23. Answer: 8.



We have

$\angle ABE = 180^\circ - 60^\circ = 120^\circ = \angle ECD$ and

$\angle AEB = 180^\circ - (\angle BAE + 120^\circ) = \angle CDE$.

Therefore ΔABE is similar to ΔECD, and it follows that AB:BE = EC:CD.
Hence $BC^2 = (BE)(EC) = (AB)(CD) = 64\ cm^2$, so BC = 8 cm.

24. Answer: 502
Since $x + 2y = 2008$, $x \cdot 2y$ has the maximum value if and only if
$x = 2y = \frac{2008}{2} = 1004$. Note that $2xy$ has the maximum value if and only if xy does. Thus $x - y = 1004 - 502 = 502$.

25. Answer: 4
Using the identity $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$ with $a = \sin^2 A$ and $b = \cos^2 A$, we have

$$\sin^6 A + \cos^6 A = (\sin^2 A + \cos^2 A)((\sin^2 A + \cos^2 A)^2 - 3\sin^2 A\cos^2 A)$$
$$= 1 - 3\sin^2 A\cos^2 A = 1 - \frac{3}{4}\sin^2 2A$$
$$= 1 - \frac{3}{4}(1 - \cos^2 2A) = 1 - \frac{3}{4}\left(1 - \frac{5}{9}\right) = \frac{2}{3}.$$

Hence $6\sin^6 A + 6\cos^6 A = 4$.

26. Answer: 153

Note that 1 is certainly the smallest factor of N, so 1 and 3 are the two smallest factors of N. Consequently, N and $\frac{N}{3}$ are the two largest factors of N. Thus we obtain the equation $N + \frac{N}{3} = 204$. Solving the equation gives $N = 153$.

27. Answer: 1100

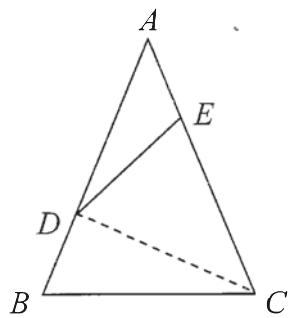
$$S = \frac{\sqrt{2}+\sqrt{1}}{\sqrt{1\times 2}} - \frac{\sqrt{3}+\sqrt{2}}{\sqrt{2\times 3}} + \frac{\sqrt{4}+\sqrt{3}}{\sqrt{3\times 4}} - \cdots + \frac{\sqrt{100}+\sqrt{99}}{\sqrt{99\times 100}}$$
$$= \left(\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}\right) - \cdots + \left(\frac{1}{\sqrt{99}}+\frac{1}{\sqrt{100}}\right)$$
$$= 1 + \frac{1}{\sqrt{100}} = 1 + \frac{1}{10} = \frac{11}{10}.$$

Hence $1000S = 1100$.

28. Answer: 2875

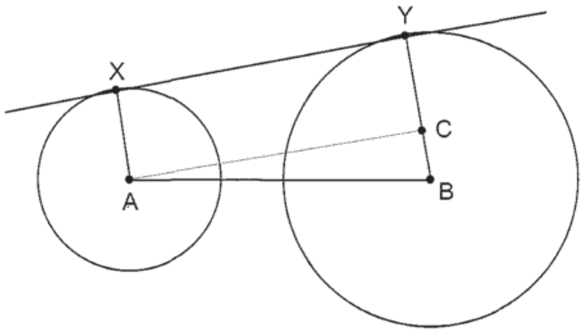
We look at the prime factorizations of 1125, 2925 and 1725. We have
$$1125 = 3^2 \times 5^3,\quad 2925 = 3^2 \times 5^2 \times 13,\ 1725 = 3 \times 5^2 \times 23.$$
Since
$$\text{LCM}(3^2 \times 5^3, 3^2 \times 5^2 \times 13, N) = \text{LCM}(3 \times 5^2 \times 23, 3^2 \times 5^2 \times 13, N),$$
we see that the least possible value of N is $5^3 \times 23 = 2875$.

29. Answer: 12



Consider ΔADE and ΔCDE. Since CE = 3AE, we see that
$$\text{area of } \Delta CDE = 3 \times \text{area of } \Delta ADE = 3x \text{ cm}^2.$$
Similarly, we have
$$\text{area of } \Delta BCD = \frac{1}{3} \times \text{area of } \Delta ACD.$$
Now area of ΔACD = area of ΔADE + area of ΔCDE = $4x$ cm^2, so
$$\text{area of } \Delta BCD = \frac{4x}{3} \text{ cm}^2.$$
Thus we obtain the equation $3x + \frac{4x}{3} = 52$, which gives $x = 12$.

30. Answer: 16



Let the radii of the circles with centres A and B be x cm and y cm respectively, and let C be the point on the line segment BY such that AC is parallel to XY. Then AC = 40 cm and BC = $(y-x)$ cm. Since ΔABC is a right-angled triangle, we have $(y-x)^2+40^2=41^2$ by Pythagorean theorem, so

$$y-x=9.$$

Now consider the trapezium ABYX. We have $300=\frac{40}{2}(x+y)$, which gives

$$x+y=15.$$

Solving the two simultaneous equations, we obtain $x = 3$ and $y = 12$. Hence

$$\frac{b}{a}=\left(\frac{12}{3}\right)^2=16.$$

31. Answer: 7

$$3\sin\left(x+\frac{\pi}{9}\right)+5\sin\left(x+\frac{4\pi}{9}\right)$$

$$=3\sin\left(x+\frac{\pi}{9}\right)+5\sin\left(x+\frac{\pi}{9}+\frac{\pi}{3}\right)$$

$$=3\sin\left(x+\frac{\pi}{9}\right)+5\sin\left(x+\frac{\pi}{9}\right)\cos\frac{\pi}{3}+5\cos\left(x+\frac{\pi}{9}\right)\sin\frac{\pi}{3}$$

$$=\frac{11}{2}\sin\left(x+\frac{\pi}{9}\right)+\frac{5\sqrt{3}}{2}\cos\left(x+\frac{\pi}{9}\right).$$

Hence the maximum value is $\sqrt{\left(\frac{11}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2}=7$.

32. Answer: 75582

We note that each 19-digit binary sequence containing exactly eight '0's and eleven '1's can be matched uniquely to such an 11-digit positive integer in the following way: Each '1' will be replaced by a digit from 1 to 9, and one more than the number of '0's to the left of a particular '1' indicates the value of the digit. For example, 0110000010101111111 is matched to 22789999999, and

1110000000011111111 is matched to 11199999999. It follows that the required number is $\binom{19}{8} = 75582$.

33. Answer: 2501

First we note that if $\sqrt{n-100}+\sqrt{n+100}$ is a rational number, then both $\sqrt{n-100}$ and $\sqrt{n+100}$ are rational. Since n is a positive integer, this implies that $\sqrt{n-100}$ and $\sqrt{n+100}$ are integers. Let $\sqrt{n-100}=k$ and $\sqrt{n+100}=\ell$. Squaring both sides of the equations, we obtain $n - 100 = k^2$ and $n + 100 = \ell^2$. This gives

$$200 = \ell^2 - k^2 = (\ell - k)(\ell + k). \quad \text{---- (1)}$$

Now for n to be the largest positive integer for which $\sqrt{n-100}+\sqrt{n+100}$ is rational, $\ell - k = \sqrt{n+100}-\sqrt{n-100}$ should be as small as possible. We rule out $\ell - k = 1$ since it leads to ℓ and k being non-integers from equation (1). Thus we have $\ell - k = 2$, and it follows from equation (1) that $\ell + k = 100$. Hence $\ell = 51$ and $k = 49$, and $n = k^2 + 100 = 49^2 + 100 = 2501$.

34. Answer: 36

We first find out which two consecutive numbers from S are not factors of N. Clearly 1 is a factor of N. Note that if an integer k is not a factor of N, then $2k$ is not a factor of N either. Therefore for $2 \le k \le 10$, since $2k$ is in S, k must be a factor of N, for otherwise, there would be at least three numbers from S (the two consecutive numbers including k, and $2k$) that are not factors of N. Hence 2, 3, …, 10 are factors of N. Then it follows that $12 = 3 \times 4$, $14 = 2 \times 7$, $15 = 3 \times 5$, $18 = 2 \times 9$, $20 = 4 \times 5$ are also factors of N. Consequently, since the two numbers from S that are not factors of N are consecutive, we deduce that 11, 13, and 19 are factors of N as well. Thus we conclude that 16 and 17 are the only two consecutive numbers from S that are not factors of N. Hence

$$N = 2^3 \times 3^2 \times 5 \times 7 \times 11 \times 13 \times 19 = 6846840,$$

so the sum of digits of $N = 2 \times (6 + 8 + 4) = 36$.

35. Answer: 3764

Let $U = \{1, 2, 3, \ldots, 74, 75\}$ be the set of integers from 1 to 75. We first find the number of integers in U that are relatively prime to $75 = 3 \times 5^2$. Let $A = \{n \in U : 3$ is a factor of $n\}$ and $B = \{n \in U : 5$ is a factor of $n\}$. Then $A \cup B$ is the set of integers in U that are not relatively prime to 75. Note that $|A| = 25$, $|B| = 15$ and $|A \cap B| = |\{ n \in U : 15$ is a factor of $n\}| = 5$. By the principle of inclusion-exclusion, $|A \cup B| = |A| + |B| - |A \cap B| = 35$. Therefore the number of integers in U that are relatively prime to 75 is $|U| - |A \cup B| = 75 - 35 = 40$. Thus

$$a_1 = 1,\ a_2 = 2,\ a_3 = 4,\ a_4 = 7,\ a_5 = 8,\ \ldots,\ a_{40} = 74.$$

Now by division algorithm, any non-negative integer can be written uniquely in the form $75k + r$ for some integers k and r, where $k \ge 0$ and $0 \le r \le 74$. Note that $\gcd(75k + r, 75) = 1$ if and only if $\gcd(r, 75) = 1$. Therefore the sequence $\{a_n\}_{n\ge1}$

can be written in the form $a_n = 75k + a_i$, where $k \geq 0$ and $1 \leq i \leq 40$. Indeed, k is given by $k = \left\lfloor \frac{n}{40} \right\rfloor$ and i is the remainder when n is divided by 40. Thus for $n =$ 2008, $k = \left\lfloor \frac{2008}{40} \right\rfloor = 50$ and $i = 8$. Hence

$$a_{2008} = 75 \times 50 + a_8 = 3750 + 14 = 3764.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Senior Section, Round 2)

Saturday, 28 June 2008 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let $ABCD$ be a trapezium with $AD \parallel BC$. Suppose K and L are, respectively, points on the sides AB and CD such that $\angle BAL = \angle CDK$. Prove that $\angle BLA = \angle CKD$.

2. Determine all primes p such that
$$5^p + 4 \cdot p^4$$
is a perfect square, i.e., the square of an integer.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that

   (i) $f(2u) = f(u+v)f(v-u) + f(u-v)f(-u-v)$ for all $u, v \in \mathbb{R}$, and

   (ii) $f(u) \geq 0$ for all $u \in \mathbb{R}$.

4. There are 11 committees in a club. Each committee has 5 members and every two committees have a member in common. Show that there is a member who belongs to 4 committees.

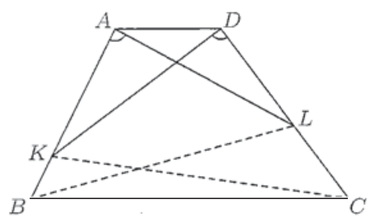
5. Let $a, b, c \geq 0$. Prove that
$$\frac{(1+a^2)(1+b^2)(1+c^2)}{(1+a)(1+b)(1+c)} \geq \frac{1}{2}(1+abc).$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Senior Section, Round 2 Solutions)

1. It's clear that $ABLK$ is cyclic. Thus $\angle ADL + \angle AKL = 180°$. Therefore $\angle BKL + \angle BCL = (180° - \angle AKL) + (180° - \angle ADL) = 180°$, so that $BCLK$ is also cyclic. Hence $\angle ABL = \angle DCK$ and $\angle BLA = 180° - \angle ABL - \angle BAL = 180° - \angle DCK - \angle CDK = \angle CKD$.



2. Let $5^p + 4 \cdot p^4 = q^2$. Then
$$5^p = (q - 2p^2)(q + 2p^2).$$
Thus
$$q - 2p^2 = 5^s, \quad q + 2p^2 = 5^t \quad \text{where } 0 \leq s < t \text{ and } s + t = p$$
Eliminating q, we get $4p^2 = 5^s(5^{t-s} - 1)$. If $s > 0$, then $5 \mid 4p^2$. Thus $p = 5$ and the given expression is indeed a square. If $s = 0$, then $t = p$ and we have $5^p = 4p^2 + 1$. We shall prove by induction that $5^k > 4k^2 + 1$ for every integer $k \geq 2$. The inequality certainly holds for $k = 2$. So we assume that it holds for some $k \geq 2$. Note that
$$\frac{4(k+1)^2 + 1}{4k^2 + 1} = \frac{4k^2 + 1}{4k^2 + 1} + \frac{8k}{4k^2 + 1} + \frac{4}{4k^2 + 1} < 1 + 1 + 1 < 5 \quad \text{for } k \geq 2.$$
Thus
$$5^{k+1} = 5 \cdot 5^k > 5(4k^2 + 1) > 4(k+1)^2 + 1.$$

3. f is either constantly 0 or constantly $1/2$.
Clearly, either of the constant functions above satisfies the requirements. Conversely, suppose the given conditions hold. Setting $u = v$, we have
$$f(2u) = f(2u)f(0) + f(0)f(-2u).$$
Case 1. $f(0) = 0$.
Then $f(2u) = 0$ for all $u \in \mathbb{R}$ and hence f is constantly 0.

Case 2. $f(0) = c \neq 0$.
Then
$$f(-2u) = \frac{1-c}{c} f(2u) \quad \text{for all } u \in \mathbb{R}.$$
Setting $u = 0$, we have in particular $c = 1 - c$. Hence $c = 1/2$. It follows that $f(-u) = f(u)$ for all $u \in \mathbb{R}$. Therefore,
$$f(2u) = f(u+v)f(u-v) + f(u-v)f(u+v) = 2f(u+v)f(u-v).$$
Setting $u = 0$, we have $1/2 = f(0) = 2f(v)f(-v) = 2(f(v))^2$. Thus $f(v) = 1/2$ for all $v \in \mathbb{R}$ since $f(v) \geq 0$.

4. Form an incidence matrix A where the rows are indexed by the committees and the columns are indexed by the members and where the (i,j) entry, $a_{ij} = 1$ if member j is in committee i. Then there are five 1's in each row and for each i, j, there exists k such that $a_{ik} = a_{jk} = 1$. Thus there are fifty five 1's in A. We need to prove that there is column with four 1's. Without loss of generality, assume that $a_{1i} = 1$ for $i = 1, \ldots, 5$. Consider the submatrix B formed by the first five columns and the last ten rows. Each row of B has at least one 1. Hence B has ten 1's. If there is a column with three 1's, then A has a column with four 1's and we are done. If not, then every column has two 1's and thus each of the corresponding columns in A has three 1's. By consider all the other rows, we see that if no column has four 1's, then every column must have exactly three 1's. But this is impossible as there are fifty five 1's in A but $3 \nmid 55$.

5. First, for any real number t, we have

$$2(1+t^2)^3 = (1+t^3)(1+t)^3 + (1-t^3)(1-t)^3 \geq (1+t^3)(1+t)^3,$$

with equality if and only if $t = 1$. In particular, for nonnegative t,

$$2^{\frac{1}{3}} \frac{1+t^2}{1+t} \geq (1+t^3)^{\frac{1}{3}},$$

with equality if and only if $t = 1$. Thus

$$\begin{aligned}
2\frac{(1+a^2)(1+b^2)(1+c^2)}{(1+a)(1+b)(1+c)} &\geq (1+a^3)^{\frac{1}{3}}(1+b^3)^{\frac{1}{3}}(1+c^3)^{\frac{1}{3}} \\
&= (1+a^3+b^3+c^3+a^3b^3+b^3c^3+c^3a^3+a^3b^3c^3)^{\frac{1}{3}} \\
&\geq (1+3abc+3a^2b^2c^2+a^3b^3c^3)^{\frac{1}{3}} \\
&\geq 1+abc.
\end{aligned}$$

Equality holds if and only if $a = b = c = 1$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Open Section, Round 1)

Wednesday, 28 May 2008 **0930-1200**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

1. Determine the number of three-element subsets of the set $\{1, 2, 3, 4, \ldots, 120\}$ for which the sum of the three elements is a multiple of 3.

2. There are 10 students taking part in a mathematics competition. After the competition, they discover that each of them solves exactly 3 problems and any 2 of them solve at least 1 common problem. What is the minimum number of students who solve a common problem which is solved by most students?

3. Evaluate the sum
$$\sum_{n=1}^{6237} \lfloor \log_2(n) \rfloor,$$
where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

4. Determine the number of positive integer divisors of 998^{49999} that are not the divisors of 998^{49998}.

5. Let $p(x)$ be a polynomial with real coefficients such that for all real x,
$$2(1 + p(x)) = p(x - 1) + p(x + 1)$$
and $p(0) = 8, p(2) = 32$. Determine the value of $p(40)$.

6. In the triangle ABC, $AC = 2BC$, $\angle C = 90°$ and D is the foot of the altitude from C onto AB. A circle with diameter AD intersects the segment AC at E. Find $AE : EC$.

7. In the triangle ABC, $AB = 8, BC = 7$ and $CA = 6$. Let E be the point on BC such that $\angle BAE = 3\angle EAC$. Find $4AE^2$.

8. In the triangle ABC, the bisectors of $\angle A$ and $\angle B$ meet at the incentre I, the extension of AI meets the circumcircle of triangle ABC at D. Let P be the foot of the perpendicular from B onto AD, and Q a point on the extension of AD such that $ID = DQ$. Determine the value of $(BQ \times IB)/(BP \times ID)$.

9. In a convex quadrilateral $ABCD$, $\angle BAC = \angle CAD$, $\angle ABC = \angle ACD$, the extensions of AD and BC meet at E, and the extensions of AB and DC meet at F. Determine the value of
$$\frac{AB \cdot DE}{BC \cdot CE}.$$

10. For any positive integer n, let N_n be the set of integers from 1 to n, i.e., $N_n = \{1, 2, 3, \cdots, n\}$. Now assume that $n \geq 10$. Determine the maximum value of n such that the following inequality
$$\min_{\substack{a,b \in A \\ a \neq b}} |a - b| \leq 10$$
holds for each $A \subseteq N_n$ with $|A| \geq 10$.

11. How many four-digit numbers greater than 5000 can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if only the digit 4 may be repeated?

12. Three girls A, B and C, and nine boys are to be lined up in a row. Let n be the number of ways this can be done if B must lie between A and C, and A, B must be separated by exactly 4 boys. Determine $\lfloor n/7! \rfloor$.

13. Determine the number of 4-element subsets $\{a, b, c, d\}$ of $\{1, 2, 3, 4, \cdots, 20\}$ such that $a + b + c + d$ is divisible by 3.

14. Find how many three digit numbers, lying between 100 and 999 inclusive, have two and only two consecutive digits identical.

15. Find the maximum natural number which are divisible by 30 and have exactly 30 different positive divisors.

16. Determine the number of 0's at the end of the value of the product $1 \times 2 \times 3 \times 4 \times \cdots \times 2008$.

17. Let a_k be the coefficient of x^k in the expansion of $(1 + 2x)^{100}$, where $0 \leq k \leq 100$. Find the number of integers $r : 0 \leq r \leq 99$ such that $a_r < a_{r+1}$.

18. Let a_k be the coefficient of x^k in the expansion of
$$(x + 1) + (x + 1)^2 + (x + 1)^3 + (x + 1)^4 + \cdots + (x + 1)^{99}.$$
Determine the value of $\lfloor a_4/a_3 \rfloor$.

19. Let $a, b. c, d, e$ be five numbers satisfying the following conditions:
$$a + b + c + d + e = 0, \quad \text{and}$$
$$abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde = 2008.$$
Find the value of $a^3 + b^3 + c^3 + d^3 + e^3$.

20. Let $a_1, a_2, \ldots$ be a sequence of rational numbers such that $a_1 = 2$ and for $n \geq 1$
$$a_{n+1} = \frac{1 + a_n}{1 - a_n}.$$
Determine $30 \times a_{2008}$.

21. Find the number of eight-digit integers comprising the eight digits from 1 to 8 such that $(i+1)$ does not immediately follow i for all i that runs from 1 to 7.

22. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ where a_0, a_1, a_2, a_3 and a_4 are constants with $a_4 \neq 0$. When divided by $x - 2003$, $x - 2004$, $x - 2005$, $x - 2006$ and $x - 2007$, respectively, $f(x)$ leaves a remainder of $24, -6, 4, -6$ and 24. Find the value of $f(2008)$.

23. Find the number of 10-letter permutations comprising 4 a's, 3 b's, 3 c's such that no two adjacent letters are identical.

24. Let $f(x) = x^3 + 3x + 1$, where x is a real number. Given that the inverse function of f exists and is given by

$$f^{-1}(x) = \left(\frac{x - a + \sqrt{x^2 - bx + c}}{2}\right)^{1/3} + \left(\frac{x - a - \sqrt{x^2 - bx + c}}{2}\right)^{1/3}$$

where a, b and c are positive constants, find the value of $a + 10b + 100c$.

25. Between 1 and 8000 inclusive, find the number of integers which are divisible by neither 14 nor 21 but divisible by either 4 or 6.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Open Section, Round 1 Solutions)

1. Answer: 93640.

   For $i = 0, 1, 2$, let $A_i = \{x \mid 1 \leq x \leq 120$ and $x \equiv i \pmod 3\}$. Then $|A_i| = 40$. If $\{a, b, c\}$ is a 3-element subset of the given set, then 3 divides $a + b + c$ if and only if exactly one of the following conditions holds: (i) all a, b, c are in A_0, or in A_1 or in A_2, (ii) one of the a, b, c is in A_0, another in A_1, and the third one in A_2. The number of 3-element subsets of A_i is $\binom{40}{3}$. For each choice of a in A_0, b in A_1 and c in A_2, we get a 3-element subset such that 3 divides $a + b + c$. Thus the total number of 3-element subsets $\{a, b, c\}$ such that 3 divides $a + b + c$ is equal to $3\binom{40}{3} + 40^3 = 93640$.

2. Answer: 5.

   Without loss of generality, we may assume that every problem is solved by some student. Let A be one of the students. By assumption each of the 9 other students solves at least one common problem as A. By the pigeonhole principle, there are at least 3 students who solve a common problem which is also solved by A. Therefore the minimum number of students who solve a common problem which is solved by most students is at least 4. If the minimum number is exactly 4, then each problem is solved by exactly 4 students. If there are n problems in the competition, then $4n = 30$. But this is a contradiction since 4 does not divide 30. Hence the minimum number of students who solve a problem which is solved by most students is at least 5. We shall show that 5 is in fact the minimum in the following example, where we write (123) to mean that a student solves problems $1, 2$ and 3:

   $$\begin{array}{ccccc} (123) & (134) & (145) & (156) & (162) \\ (235) & (245) & (246) & (346) & (356) \end{array}$$

3. Answer: 66666.

   Note that $\lfloor \log_2(1) \rfloor = 0$, and $\lfloor \log_2(n) \rfloor = k$ if $2^k \leq n < 2^{k+1}$. Since $6237 = 2^{12} + 2141$, we have $\lfloor \log_2(n) \rfloor = 12$ for $2^{12} \leq n \leq 6237$, and there are 2142 such n's. Thus

   $$\begin{aligned} \textstyle\sum_{n=1}^{6237} \lfloor \log_2(n) \rfloor &= 0 + 1(2^2 - 2) + 2(2^3 - 2^2) + \cdots + 11(2^{12} - 2^{11}) + (12)(2142) \\ &= (11)2^{12} - (2 + 2^2 + \cdots + 2^{11}) + (12)(2142) \\ &= (11)2^{12} - 2(2^{11} - 1) + (12)(2142) \\ &= (20)2^{11} + (12)(2142) + 2 = 66666. \end{aligned}$$

4. Answer: 99999.

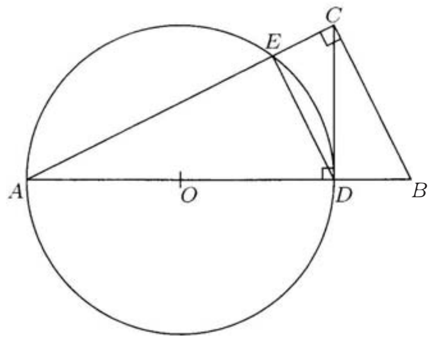
   Note that $998 = 2 \times 499$, and 499 is a prime. Any divisor of 998^{49999} has the form $d = 2^a 499^b$, where a and b are positive integers between 0 and 49999. This divisor d does not divide 998^{49998} only in two cases which are either $a = 49999$ or $b = 49999$. In the first case, d can be $2^{49999}, 2^{49999}499, 2^{49999}499^2, \ldots, 2^{49999}499^{49999}$. In the second case, d can be $499^{49999}, 499^{49999}2, 499^{49999}2^2, \ldots, 499^{49999}2^{49999}$. In each case, there are 50000 possible values, but the number $2^{49999}499^{49999}$ is counted twice. Thus the total number of required divisors is $2 \times 50000 - 1 = 99999$.

5. Answer: 2008.

   Let $p(x) = q(x) + x^2$. Substituting this into the given functional equation, we get $q(x) - q(x-1) = q(x+1) - q(x)$ for all real x. Since $q(x)$ is a polynomial, we must have $q(x) - q(x-1) \equiv b$, where b is a real constant. (The polynomial $Q(x) \equiv q(x) - q(x-1)$ can't take the same value at an infinite number of distinct points as it is eventually monotonic, unless it is a constant polynomial.) Next let $q(x) = r(x) + bx$. Substituting this into $q(x) - q(x-1) = b$, we get $r(x) = r(x-1)$ so that $r(x) \equiv c$, where c is a real constant. Therefore, $p(x) = x^2 + bx + c$. It can be easily verified that any polynomial $p(x) = x^2 + bx + c$ satisfies the given function equation. Using $p(0) = 8$ and $p(2) = 32$, we get $p(x) = x^2 + 10x + 8$. Thus $p(40) = 2008$.

6. Answer: 4.

   Since AD is a diameter, we have $\angle AED = 90°$ so that DE is parallel to BC.



   As $\triangle ADC$ is similar to $\triangle ACB$, we have $AC/AD = AB/AC$ so that $AC^2 = AD \times AB$. Similarly, $BC^2 = AB \times BD$. Thus $AE/EC = AD/BD = (AC/BC)^2 = 4$.

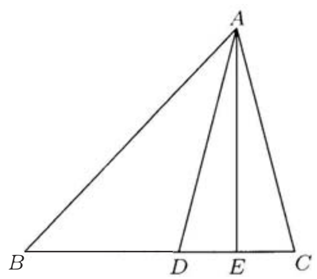
7. Answer: 135.

   In general, we let $AB = c, AC = b$ and $BC = a$. Let AD be the bisector of $\angle A$. Using the angle-bisector theorem, $BD/CD = c/b$. Thus $BD = ac/(b+c)$ and $CD = ab/(b+c)$. By Stewart's theorem,

$$(\frac{ab}{b+c})c^2 + (\frac{ac}{b+c})b^2 = aAD^2 + \frac{abc}{(b+c)^2}a^2,$$

   which after solving for AD^2 gives

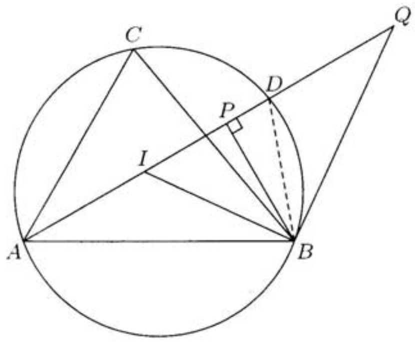
$$AD^2 = bc\left[1 - (\frac{a}{b+c})^2\right].$$



Substituting the values of $a = 7, b = 6, c = 8$, we get $AD = 6$. Thus $\triangle ACD$ is isosceles and the bisector AE of $\angle CAD$ is perpendicular to BC and E is the midpoint of CD. As $DC = 3$, we have $EC = 3/2$. Using Pythagoras' theorem, we get $AE^2 = AC^2 - EC^2 = 6^2 - (\frac{3}{2})^2 = 135/4$. Therefore, $4AE^2 = 135$.

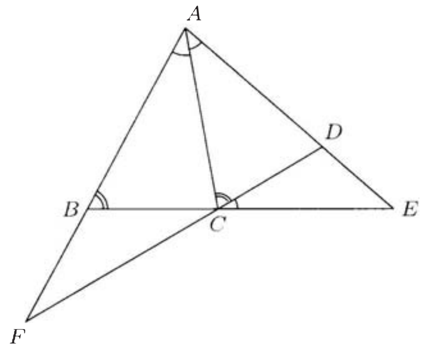
8. Answer: 2.

   Let's prove that $\triangle BPQ$ is similar to $\triangle IBQ$.



   First $\angle IAB = \angle CAD = \angle CBD$. As $\angle IBA = \angle IBC$, we have $\angle IAB + \angle IBA = \angle CBD + \angle IBC$ so that $\angle DIB = \angle DBI$, thus $DI = DB = DQ$. This means $\triangle IBQ$ is a right-angled triangle with $\angle IBQ = 90°$. As $\angle Q$ is a common angle, we thus have $\triangle BPQ$ is similar to $\triangle IBQ$. Hence, $BQ/BP = IQ/IB = 2ID/IB$. Consequently, $(BQ \times IB)/(BP \times ID) = 2$.

9. Answer: 1.



   Since $\angle ACE$ is an exterior angle for the triangle ABC, we have $\angle ACE = \angle ABC + \angle BAC$, hence $\angle ACE = \angle ACD + \angle CAD$. It follows that $\angle CAD = \angle DCE$, so the triangles CED and AEC are similar. For this, we obtain $CE/AE = DE/CE$. But $\angle BAC = \angle CAE$, so we get $BC/CE = AB/AE$. Using these equalities, we get $(AB \cdot DE)/(BC \cdot CE) = 1$.

10. Answer: 99.

    First assume that $n \geq 100$. Consider the following set A:

$$A = \{1 + 11i : i = 0, 1, 2, \cdots, 9\}.$$

Note that $|A| = 10$, $A \subseteq N_n$ and

$$\min_{\substack{a,b\in A\\ a\neq b}} |a-b| = 11.$$

Hence the answer is not larger than 99.

Now consider the case that $n \leq 99$. For $i = 1, 2, 3, \cdots, 9$,

$$P_i = \{11(i-1) + j : 1 \leq j \leq 11\}.$$

Note that

$$N_n \subseteq \bigcup_{1\leq i\leq 9} P_i.$$

Now assume that A is any subset of N_n with $|A| \geq 10$. By Pigeonhole Principle, $|A \cap P_i| \geq 2$ for some $i : 1 \leq i \leq 9$. Let $c_1, c_2 \in A \cap P_i$. Thus

$$\min_{\substack{a,b\in A\\ a\neq b}} |a-b| \leq |c_1 - c_2| \leq (11(i-1)+11) - (11(i-1)+1) = 10.$$

11. Answer: 2645.

    Let $\overline{abcd}$ represent the integer $a \times 10^3 + b \times 10^2 + c \times 10 + d$.

    Note that $\overline{abcd} > 5000$ iff $a \geq 5$ and b, c, d are not all 0. a must be a number in $\{5, 6, 7, 8, 9\}$. Suppose that a is selected from $\{5, 6, 7, 8, 9\}$.

    If 4 is not repeated, then the number of ways to choose b, c, d is $9 \times 8 \times 7$;

    If 4 appears exactly twice, then the number of ways to choose b, c, d is $\binom{3}{2} \times 8$;

    If 4 appears exactly three times, then the number of ways to choose b, c, d is 1.

    Hence the answer is

    $$5 \times (9 \times 8 \times 7 + \binom{3}{2} \times 8 + 1) = 2645.$$

12. Answer: 3024.

    Let Φ be the set of arrangements of these girls and boys under the condition that A, B must be separated by exactly 4 boys. Form a block P with A, B at the two ends and exactly 4 boys between them. The number of ways to form such a block P is

    $$2 \times \binom{9}{4} \times 4!.$$

    Then we have

    $$|\Phi| = 2 \times \binom{9}{4} \times 4! \times 7!,$$

    as we can consider such a block P as one item, and there are still 5 boys and one girl (i.e., C).

    Note that in any arrangement in Φ, C is outside the block between A and B. In exactly half of the arrangements of Φ, B is between A and C. Hence

    $$n = |\Phi|/2 = \binom{9}{4} \times 4! \times 7!.$$

    Hence the answer is $\binom{9}{4} \times 4! = 3024$.

13. Answer: 11901.

For $i = 0, 1, 2$, let
$$A_i = \{j : 1 \leq j \leq 20, j \equiv i \pmod{\text{od } 3} \}.$$
Note that $|A_0| = 6$, $|A_1| = 7$ and $|A_2| = 7$.

It can be shown that for $a, b, c, d \in \{1, 2, 3, 4, \cdots, 20\}$, $a + b + c + d$ is divisible by 3 iff

(i) $|\{a, b, c, d\} \cap A_i| = 2$ for all $i = 1, 2$; or

(ii) $|\{a, b, c, d\} \cap A_0| = 1$ and $|\{a, b, c, d\} \cap A_i| = 3$ for some $i : 1 \leq i \leq 2$; or

(iii) $|\{a, b, c, d\} \cap A_0| = 2$ and $|\{a, b, c, d\} \cap A_i| = 1$ for all $i = 1, 2$; or

(iv) $|\{a, b, c, d\} \cap A_0| = 4$.

Thus the number of 4-element subsets $\{a, b, c, d\}$ of $\{1, 2, 3, 4, \cdots, 20\}$ such that $a+b+c+d$ is divisible by 3 is
$$\binom{|A_1|}{2}\binom{|A_2|}{2} + \binom{|A_0|}{1}\binom{|A_1|}{3} + \binom{|A_0|}{1}\binom{|A_2|}{3} + \binom{|A_0|}{2} \times |A_1| \times |A_2| + \binom{|A_0|}{4}$$
$$= \binom{7}{2}^2 + 6\binom{7}{3} \times 2 + \binom{6}{2}^2 \times 7^2 + \binom{6}{4} = 11901.$$

14. Answer: 162.

There are two possible formats for three digit numbers to have two and only two consecutive digits identical:

(i) $\overline{aac}$ where $a \neq 0$ and $c \neq a$, or

(ii) $\overline{abb}$ where $a \neq 0$ and $a \neq b$.

Thus the number of such integers is
$$9 \times 9 + 9 \times 9 = 162.$$

15. Answer: 11250.

Let a be a natural number divisible by 30. Thus a can be expressed as
$$a = 2^{n_2} 3^{n_3} 5^{n_5} \prod_{1 \leq i \leq r} p_i^{k_i},$$
where $r \geq 0$, $k_i \geq 0$, and n_2, n_3, n_5 are positive integers. The number of positive divisors of a is
$$(n_2 + 1)(n_3 + 1)(n_5 + 1) \prod_{1 \leq i \leq r} (k_i + 1).$$
Since $30 = 2 \times 3 \times 5$, we have $r = 0$ if
$$(n_2 + 1)(n_3 + 1)(n_5 + 1) \prod_{1 \leq i \leq r} (k_i + 1) = 30.$$
Hence $30|a$ and a has exactly 30 positive divisors iff $a = 2^{n_2} 3^{n_3} 5^{n_5}$ and $(n_2 + 1)(n_3 + 1)(n_5 + 1) = 30$, where n_2, n_3, n_5 are all positive. Further, $(n_2+1)(n_3+1)(n_5+1) = 30$, where n_2, n_3, n_5 are all positive, iff $\{n_2, n_3, n_5\} = \{1, 2, 4\}$.

The maximum value of such a is $a = 2 \times 3^2 \times 5^4 = 11250$.

16. Answer: 500.

    Let m be the maximum integer such that 2^m is a factor of $1 \times 2 \times 3 \times 4 \times \cdots \times 2008$, and n the maximum integer such that 5^n is a factor of $1 \times 2 \times 3 \times 4 \times \cdots \times 2008$.

    Then the number of 0's at the end of $1 \times 2 \times 3 \times 4 \times \cdots \times 2008$ is equal to i n $\{m, n\}$. It is obvious that $m \geq n$.

    Observe that $2008 < 5^5$. For $1 \leq k \leq 4$, the number, denoted by a_k, of integers in $\{1, 2, 3, 4, \cdots, 2008\}$, denoted by A, divisible by 5^k is

    $$a_k = \left\lfloor 2008/5^k \right\rfloor .$$

    So $a_1 = 401$, $a_2 = 80$, $a_3 = 16$, $a_4 = 3$. Thus, there are exactly 3 numbers in A divisible by 5^4;

    exactly $16 - 3 = 13$ numbers in A divisible by 5^3 but not by 5^4;

    exactly $80 - 16 = 64$ numbers in A divisible by 5^2 but not by 5^3;

    exactly $401 - 80 = 321$ numbers in A divisible by 5 but not by 5^2.

    Hence, the maximum integer n such that 5^n is a factor of $1 \times 2 \times 3 \times 4 \times \cdots \times 2008$ is

    $$n = 4 \times 3 + 3 \times 13 + 2 \times 64 + 1 \times 321 = 500.$$

17. Answer: 67.

    Let a_k be the coefficient of x^k in the expansion of $(1 + 2x)^{100}$. Then

    $$a_k = 2^k \binom{100}{k}.$$

    Thus

    $$\frac{a_{r+1}}{a_r} = 2\binom{100}{r+1}\Big/\binom{100}{r} = \frac{2(100-r)}{r+1}.$$

    Solving the inequality

    $$\frac{2(100-r)}{r+1} > 0$$

    gives the solution $r < 199/3$. It implies that

    $$a_0 < a_1 < \cdots < a_{66} < a_{67} \geq a_{68} \geq \cdots \geq a_{100}.$$

    Hence the answer is 67.

18. Answer: 19.

    Note that

    $$a_k = \binom{1}{k} + \binom{2}{k} + \binom{3}{k} + \binom{4}{k} + \cdots + \binom{99}{k} = \binom{100}{k+1}.$$

    Thus

    $$a_4/a_3 = \frac{\binom{100}{5}}{\binom{100}{4}} = \frac{96}{5}.$$

    Thus the answer is 19.

19. Answer: 6024.

Note that $(a+b+c+d+e)^3 = a^3+b^3+c^3+d^3+e^3+3a^2(b+c+d+e)+3b^2(c+d+e+a)+3c^2(d+e+a+b)+3d^2(e+a+b+c)+3e^2(a+b+c+d)+6(abc+abd+abe+\cdots)$. Since $a+b+c+d+e=0, b+c+d+e=-a$, so $3a^2(b+c+d+e)=-3a^3$. Similarly, we get $-3b^3, -3c^3, -3d^3$ and $-3e^3$. Thus, we have $2(a^3+b^3+c^3+d^3+e^3) = 6(abc+abd+abe+\cdots)$, so $a^3+b^3+c^3+d^3+e^3 = 3\times 2008 = 6024$.

20. Answer: 10.

We have

$$\begin{aligned} a_{n+1} &= \tfrac{1+a_n}{1-a_n} \\ a_{n+2} &= \tfrac{1+a_{n+1}}{1-a_{n+1}} = \tfrac{1+\frac{1+a_n}{1-a_n}}{1-\frac{1+a_n}{1-a_n}} = -\frac{1}{a_n} \\ a_{n+4} &= -\frac{1}{a_{n+2}} = a_n. \end{aligned}$$

Thus $a_{2008} = a_{2004} = \cdots = a_4 = \frac{1}{3}$.

21. Answer: 16687.

We may proceed by using the principle of inclusion and exclusion as follows. Let the universal set S be the set of 8-digit integers comprising the 8 digits from 1 to 8. Let $P_i (1 \le i \le 7)$ be the property that $(i+1)$ immediately follows i in an element of S. Hence $W(0) = 8!$, $W(i) = \binom{7}{i}(8-i)!$ for $1 \le i \le 7$. Then the answer $= E(0) = W(0) - W(1) + W(2) - \cdots - W(7) = 16687$. The answer can also be obtained from either one of the two expressions, namely, $\frac{1}{8}D_9$ or $D_8 + D_7$, where D_n is the number of derangements of the integers from 1 to n and $D_n = n![1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}]$.

22. Answer: 274.

$$\begin{aligned} f(x) = \; & \frac{(x-2004)(x-2005)(x-2006)(x-2007)(24)}{(-1)(-2)(-3)(-4)} \\ & + \frac{(x-2003)(x-2005)(x-2006)(x-2007)(-6)}{(1)(-1)(-2)(-3)} \\ & + \frac{(x-2003)(x-2004)(x-2006)(x-2007)(4)}{(2)(1)(-1)(-2)} \\ & + \frac{(x-2003)(x-2004)(x-2005)(x-2007)(-6)}{(3)(2)(1)(-1)} \\ & + \frac{(x-2003)(x-2004)(x-2005)(x-2006)(24)}{(4)(3)(2)(1)}. \end{aligned}$$

Therefore $f(2008) = 274$.

23. Answer: 248.

We shall use the *RP hex function* (where RP stands for Roger Poh) to deal with this question. Let $\#(p,q,r)$ denote the number of permutations of p copies of A, q copies of B and r copies of C such that no two adjacent letters are identical. Among these $\#(p,q,r)$ such permutations, let $\#(p;q,r)$ denote the number of those permutations which do not begin with A. Hence, we have

$$\begin{aligned} \#(p,q,r) &= \#(p-1;q,r) + \#(q-1;p,r) + \#(r-1;p,q) \quad \text{and} \\ \#(p;q,r) &= \#(q-1;p,r) + \#(r-1;p,q). \end{aligned}$$

For this question, we are required to evaluate $\#(4,3,3)$.

$$
\begin{aligned}
\#(4,3,3) &= \#(3;3,3) + 2\#(2;4,3) \\
\#(3;3,3) &= 2\#(2;3,3) = 4\#(2;2,3) = 116 \\
\#(2;2,3) &= \#(1;2,3) + \#(2;2,2) = 9 + 20 = 29 \\
\#(1;2,3) &= \#(1;1,3) + \#(2;1,2) = 2 + 7 = 9 \\
\#(1;1,3) &= \#(1,3) + \#(2;1,1) = \#(2;1,1) = 2 \\
\#(2;1,1) &= 2\#(2,1) = 2 \\
\#(2;1,2) &= \#(2,2) + \#(1;2,1) = 2 + \#(1;2,1) = 7 \\
\#(1;2,1) &= \#(1;1,1) + \#(1,2) = 2\#(1,1) + 1 = 5 \\
\#(2;2,2) &= 2\#(1;2,2) = 4\#(1;2,1) = 20 \\
\#(2;4,3) &= \#(3;2,3) + \#(2;4,2) = 45 + 21 = 66 \\
\#(3;2,3) &= \#(1;3,3) + \#(2;3,2) = 2\#(2;1,3) + \#(2;3,2) \\
\#(2;1,3) &= \#(2,3) + \#(2;1,2) = 1 + 7 = 8 \\
\#(3;2,3) &= 16 + 29 = 45 \\
\#(2;4,2) &= \#(3;2,2) + \#(1;2,4) = 2\#(1;3,2) + \#(1;2,4) \\
&= 18 + \#(1;2,4) = 18 + 3 = 21 \\
\#(4;3,3) &= 116 + 2(66) = 248.
\end{aligned}
$$

24. Answer: 521.

Let $y = x^3 + 3x + 1$. Then $x^3 + 3x + (1-y) = 0$. Let A, B be constants such that

$$(x^3 + A^3 + B^3) - 3ABx \equiv x^3 + 3x + (1-y).$$

Hence

$$A^3 + B^3 = 1 - y \qquad (1)$$

$$-3AB = 3 \qquad (2)$$

By (2), we have $B = -1/A$. Substitute this into (1) and simplify,

$$A^6 + (y-1)A^3 - 1 = 0.$$

Hence

$$A^3 = \frac{1 - y \pm \sqrt{(y-1)^2 + 4}}{2} = \frac{1 - y \pm \sqrt{y^2 - 2y + 5}}{2}.$$

Therefore $B = \left(\frac{1-y\mp\sqrt{y^2-2y+5}}{2}\right)^{1/3}$. Hence

$$
\begin{aligned}
x &= -A - B = \left(\tfrac{y-1+\sqrt{y^2-2y+5}}{2}\right)^{1/3} + \left(\tfrac{y-1-\sqrt{y^2-2y+5}}{2}\right)^{1/3} \\
&\equiv \left(\tfrac{y-a+\sqrt{y^2-by+c}}{2}\right)^{1/3} + \left(\tfrac{y-a-\sqrt{y^2-by+c}}{2}\right)^{1/3}
\end{aligned}
$$

Thus $a = 1, b = 2, c = 5$ and $a + 10b + 100c = 521$.

25. Answer: 2287.

Let $S = \{1, \ldots, 8000\}$, $A = \{x \in S : 4 \mid x\}$, $B = \{x \in S : 6 \mid x\}$, $C = \{x \in S : 14 \mid x\}$, $D = \{x \in S : 21 \mid x\}$.

$$\begin{aligned}
& |(A \cup B) \cap (C \cup D)'| \\
= \; & |A \cup B| - |(A \cup B) \cap (C \cup D)| \\
= \; & |A| + |B| - |A \cap B| - |A \cap C) \cup (B \cap C) \cup (A \cap D) \cup (B \cap D)| \\
= \; & |A| + |B| - |A \cap B| - |A \cap C| - |B \cap C| - |A \cap D| - |B \cap D| \\
& + |A \cap B \cap C| + |A \cap C \cap D| + |A \cap B \cap C \cap D| + |A \cap B \cap C \cap D| \\
& + |B \cap C \cap D| + |A \cap B \cap D| - |A \cap B \cap C \cap D| - |A \cap B \cap C \cap D| \\
& - |A \cap B \cap C \cap D| - |A \cap B \cap C \cap D| + |A \cap B \cap C \cap D| \\
= \; & \lfloor 8000/4 \rfloor + \lfloor 8000/6 \rfloor - \lfloor 8000/12 \rfloor - \lfloor 8000/28 \rfloor - \lfloor 8000/42 \rfloor - \lfloor 8000/84 \rfloor \\
& - \lfloor 8000/42 \rfloor + 3 \times \lfloor 8000/84 \rfloor + \lfloor 8000/42 \rfloor - \lfloor 8000/84 \rfloor \\
= \; & 2287.
\end{aligned}$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Open Section, Round 2)

Saturday, 5 July 2008 **0900-1330**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Find all pairs of positive integers $(n.k)$ so that $(n+1)^k - 1 = n!$.

2. In the acute triangle ABC, M is a point in the interior of the segment AC and N is a point on the extension of the segment AC such that $MN = AC$. Let D and E be the feet of the perpendiculars from M and N onto the lines BC and AB respectively. Prove that the orthocentre of $\triangle ABC$ lies on the cicumcircle of $\triangle BED$.

3. Let n, m be positive integers with $m > n \geq 5$ and with m depending on n. Consider the sequence $a_1, a_2, \ldots, a_m$ where

$$\begin{aligned} a_i &= i && \text{for } i = 1, \ldots, n \\ a_{n+j} &= a_{3j} + a_{3j-1} + a_{3j-2} && \text{for } j = 1, \ldots, m-n \end{aligned}$$

with $m - 3(m-n) = 1$ or 2, i.e., $a_m = a_{m-k} + a_{m-k-1} + a_{m-k-2}$ where $k = 1$ or 2. (Thus if $n = 5$, the sequence is $1, 2, 3, 4, 5, 6, 15$ and if $n = 8$, the sequence is $1, 2, 3, 4, 5, 6, 7, 8, 6, 15, 21$.) Find $S = a_1 + a_2 + \cdots + a_m$ if (i) $n = 2007$, (ii) $n = 2008$.

4 Let $0 < a, b < \pi/2$. Show that

$$\frac{5}{\cos^2 a} + \frac{5}{\sin^2 a \sin^2 b \cos^2 b} \geq 27\cos a + 36\sin a.$$

5. Consider a 2008×2008 chess board. Let M be the smallest number of rectangles that can be drawn on the chess board so that the sides of every cell of the board is contained in the sides of one of the rectangles. Find the value of M. (For example, for the 2×3 chess board, the value of M is 3.)

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Singapore Mathematical Olympiad (SMO) 2008

(Open Section, Round 2 Solutions)

1. If p is a prime factor of $n+1$, it is also a factor of $n!+1$. However, none of the numbers $1, \ldots, n$ is a factor of $n!+1$. So we must have $p = n+1$, i.e., $n+1$ must be prime. For $n+1 = 2, 3, 5$, we have solutions $(n,k) = (1,1), (2,1)$ and $(4,2)$. Let us show that there are no other solutions. Indeed, suppose that $n+1$ is a prime number ≥ 7. Then $n = 2m$, $m > 2$. Now $2 < m < n$ and thus $n^2 = 2mn$ divides $n!$. Thus n^2 divides

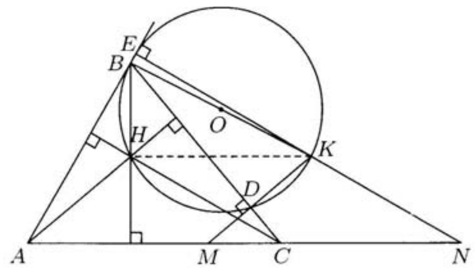
$$(n+1)^k - 1 = n^k + kn^{k-1} + \cdots + \frac{k(k-1)}{2}n^2 + nk.$$

Thus $n^2 \mid kn$ and, in particular $k \geq n$. It follows that

$$n! = (n+1)^k - 1 > n^k \geq n^n \geq n!,$$

a contradiction.

2. Let K be the point of intersection of MD and NE. It is easy to see that the circle with diameter BK is the circumcircle of $\triangle BED$. As AH is parallel to MK and CH is parallel to NK, we have $\angle HAC = \angle KMN$ and $\angle ACH = \angle MNK$. Since $AC = MN$, we thus have $\triangle AHC$ is congruent to $\triangle MKN$.



Therefore the distance from K onto AC equals to the distance from H onto AC. But H and K are on the same side with respect to the line AC, it follows that HK is parallel to AC. Therefore HK is perpendicular to BH and H lies on the circle with diameter BK circumscribing about $\triangle BED$.

3. We shall solve the problem for general n. Initially, let $a_1, a_2, \ldots, a_n$ be the *active* sequence. An *operation* removes the first three terms and append their sum as the last term of the sequence. Thus after one operation on the initial sequence, the active sequence is $a_4, a_5, \ldots, a_{n+1}$. Such operations will stop when the active sequence is of length at most 2. In this case, the last term of the active sequence is a_m. The following observations are obvious.

1. After each operation, the length of the active sequence decreases by 2 and the sum of its terms is preserved.
2. If the length of an active sequence is $3k$ and the last term is a_p, then after k operations, the length of the active sequence is k and the first term is a_{p+1}.

To compute S, we have two cases:

Case (i). n odd. Let k and r be integers such that $n = 3^k + 2r < 3^{k+1}$, i.e., $0 \le r < 3^k$. Note that $3r < n$. The first $3r$ terms, $a_1, \ldots, a_{3r}$ is called the *first block* and the sum of its terms is $M = 1 + 2 + \cdots + 3r = 3r(3r+1)/2$. Now apply the above observations. The sum of the terms of any active sequence is $N = 1 + 2 + \cdots + n = n(n+1)/2$. After r operations, the length of the active sequence is 3^k and its first term is a_{3r+1}. This active sequence is called the *second block*. After 3^{k-1} operations, the length of the active sequence is 3^{k-1}. This active sequence is called the *third block*. Repeat this until the length of the active sequence is 1. This is now the $(k+2)^{\text{nd}}$ block. Thus we have a block whose sum is M and $k+1$ blocks whose sum is N. Hence $S = M + (k+1)N$. Since $3^6 = 729$, $n = 2007 = 3^6 + 2(639)$, we have $S = 15943599$.

Case (ii). n even. Let k and r be integers such that $n = 2(3^k) + 2r < 2(3^{k+1})$, i.e., $0 \le r < 2(3^k)$. Note that $3r < n$. The first $3r$ terms, $a_1, \ldots, a_{3r}$ is called the *first block* and the sum of its terms is M. As in case i, after r operations, the length of the active sequence is $2(3^k)$ and its first term is a_{3r+1}. This active sequence is called the *second block*. After $2(3^{k-1)}$ operations, the length of the active sequence is $2(3^{k-1})$. This active sequence is called the *third block*. Repeat this until the length of the active sequence is 2. This is now the $(k+2)^{\text{nd}}$ block. Thus we have a block whose sum is M and $k+1$ blocks whose sum is N. Hence $S = M + (k+1)N$. Since $3^6 = 729$, $2008 = 2(3^6) + 2(275)$. Thus $r = 275$ and $k = 6$ and so $S = 14459977$.

4. First note that by AM-GM,

$$\frac{\cos^2 a}{\sin^2 a \sin^2 b \cos^2 b} + \frac{\sin^2 a}{\cos^2 a} \ge \frac{2}{\sin b \cos b}.$$

Thus

$$\begin{aligned}
\text{LHS} &= \left(\tfrac{5}{\cos^2 a} + \tfrac{5}{\sin^2 a \sin^2 b \cos^2 b}\right)(\cos^2 a + \sin^2 a) \\
&= 5 + 5\left(\tfrac{\cos^2 a}{\sin^2 a \sin^2 b \cos^2 b} + \tfrac{\sin^2 a}{\cos^2 a}\right) + \tfrac{5}{\sin^2 b \cos^2 b} \\
&\ge 5 + \tfrac{10}{\sin b \cos b} + \tfrac{5}{\sin^2 b \cos^2 b} \ge 5\left(1 + \tfrac{1}{\sin b \cos b}\right)^2 \\
&\ge 5\left(1 + \tfrac{2}{\sin 2b}\right)^2 \ge 45 \ge 27\cos a + 36 \sin a.
\end{aligned}$$

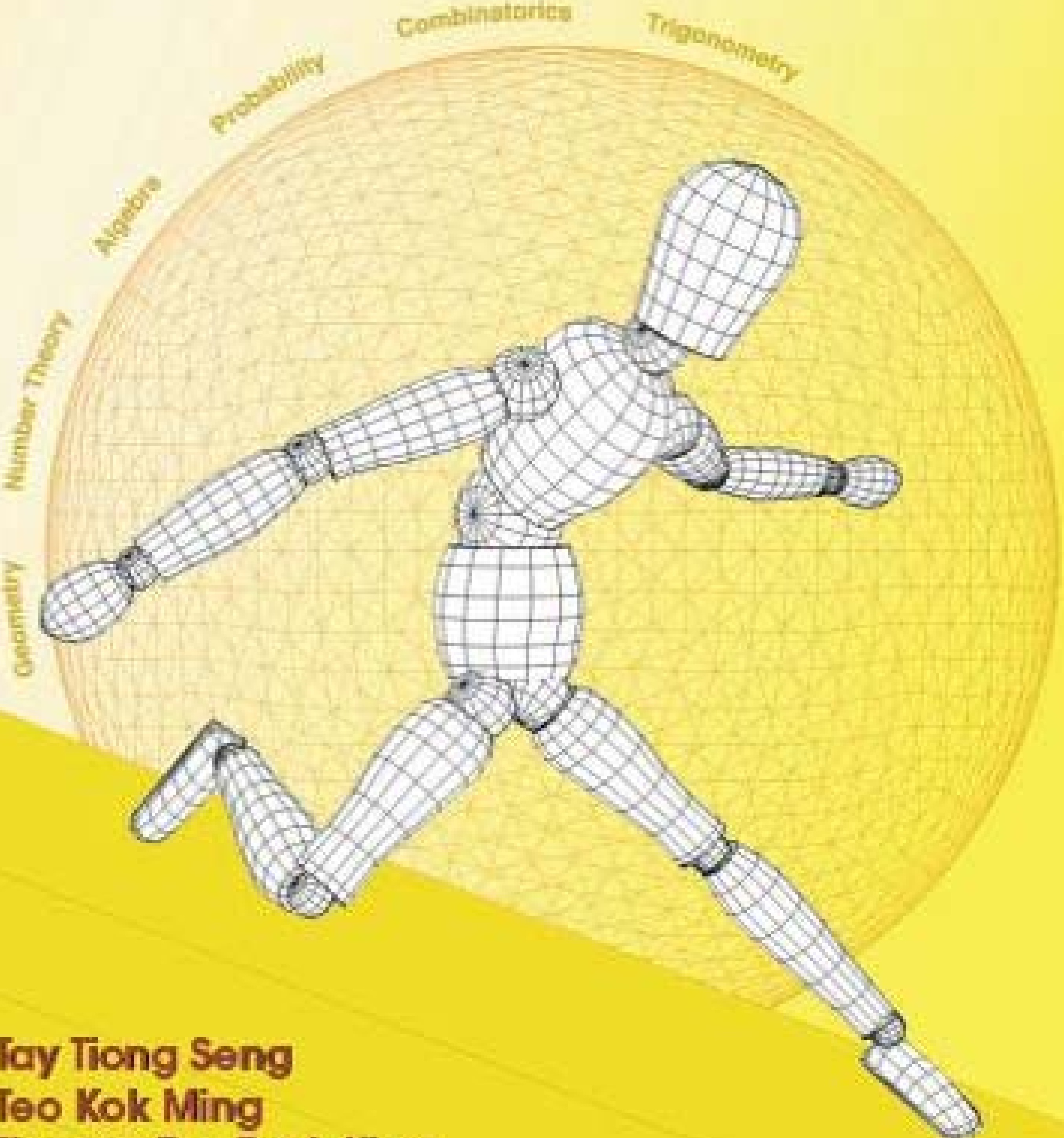
The last inequality follows because if $\sin x = 3/5$, then $\cos x = 4/5$ and $1 \ge \sin(a+x) = \sin x \cos a + \cos x \sin a$.

5. The answer is $M = 2009$. All the horizontal sides can be covered by 1004 pieces of 1×2008 rectangles except the boundary of the chess board which can be covered by the boundary rectangle. The remaining vertical sides can be covered by 1004 pieces of 2008×1 rectangles. Thus $M \leq 2009$.

   Now suppose that the chess board has been covered by M rectangles in the desired way. Let a of the rectangles have their top and bottom on the top and bottom of the board, b of the rectangles have their top on the top of the board, c of the rectangles have their bottom on the bottom of the board and d of the rectangles have neither their top nor bottom on the top or bottom of the board.

   Since there are 2007 internal horizontal lines, we have $b + c + 2d \geq 2007$ Since there are 2009 vertical lines intersecting the top of the board, we have $2a+2b \geq 2009 \quad \text{or} \quad a+b \geq 1005$. Similarly, $a + c \geq 1005$. Thus $2a + b + c \geq 2010$. Hence $2(a + b + c + d) \geq 4017 \quad \text{i.e.,} \quad M = a + b + c + d \geq 2009$.

SINGAPORE MATHEMATICAL OLYMPIADS
2009



Tay Tiong Seng
Teo Kok Ming
Thomas Teo Teck Kian
Toh Tin Lam

Published by Singapore Mathematical Society

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Junior Section)

Tuesday, 2 June 2009 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answers on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer in the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

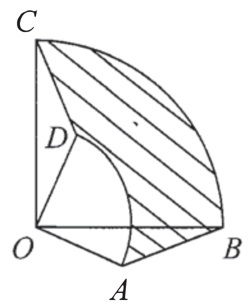
PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1 Let C_1 and C_2 be distinct circles of radius 7 cm that are in the same plane and tangent to each other. Find the number of circles of radius 26 cm in this plane that are tangent to both C_1 and C_2.

(A) 2
(B) 4
(C) 6
(D) 8
(E) none of the above

2 In the diagram below, the radius of quadrant OAD is 4 and the radius of quadrant OBC is 8. Given that $\angle COD = 30°$, find the area of the shaded region $ABCD$.



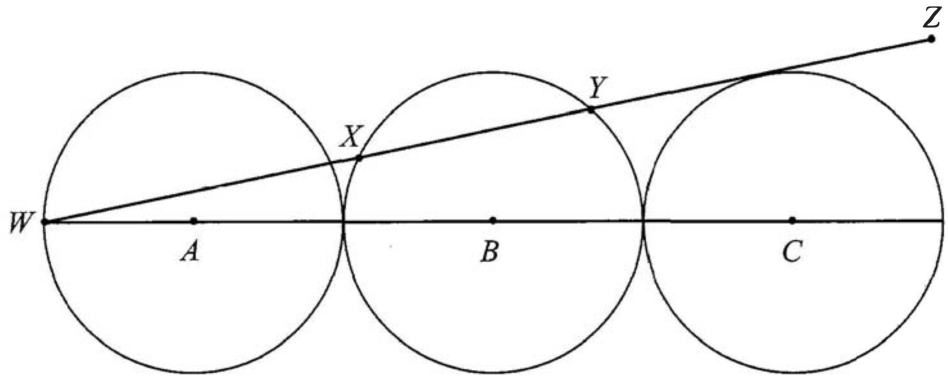
(A) 12π
(B) 13π
(C) 15π
(D) 16π
(E) none of the above

3 Let k be a real number. Find the maximum value of k such that the following inequality holds:

$$\sqrt{x-2} + \sqrt{7-x} \geq k.$$

(A) $\sqrt{5}$
(B) 3
(C) $\sqrt{2} + \sqrt{3}$
(D) $\sqrt{10}$
(E) $2\sqrt{3}$

4 Three circles of radius 20 are arranged with their respective centres A, B and C in a row. If the line WZ is tangent to the third circle, find the length of XY.



(A) 30
(B) 32
(C) 34
(D) 36
(E) 38

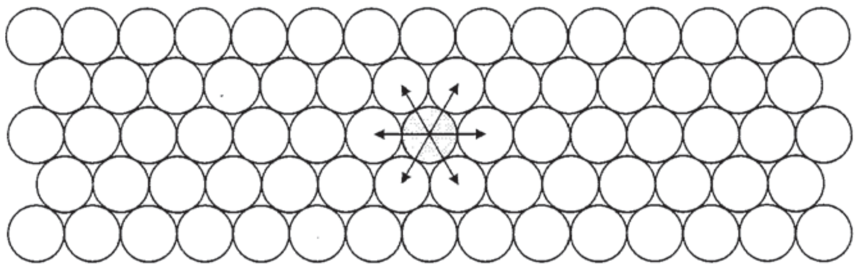
5 Given that x and y are both negative integers satisfying the equation $y = \dfrac{10x}{10-x}$, find the maximum value of y.

(A) -10
(B) -9
(C) -6
(D) -5
(E) None of the above

6 The sequence a_n satisfy $a_n = a_{n-1} + n^2$ and $a_0 = 2009$. Find a_{50}.

(A) 42434
(B) 42925
(C) 44934
(D) 45029
(E) 45359

7 Coins of the same size are arranged on a very large table (the infinite plane) such that each coin touches six other coins. Find the percentage of the plane that is covered by the coins.



(A) $\dfrac{20}{\sqrt{3}}\pi\ \%$
(B) $\dfrac{50}{\sqrt{3}}\pi\ \%$
(C) $16\sqrt{3}\,\pi\ \%$
(D) $17\sqrt{3}\,\pi\ \%$
(E) $18\sqrt{3}\,\pi\ \%$

8 Given that x and y are real numbers satisfying the following equations:

$$x + xy + y = 2 + 3\sqrt{2} \quad \text{and} \quad x^2 + y^2 = 6,$$

find the value of $|x + y + 1|$.

(A) $1 + \sqrt{3}$
(B) $2 - \sqrt{3}$
(C) $2 + \sqrt{3}$
(D) $3 - \sqrt{2}$
(E) $3 + \sqrt{2}$

9 Given that $y = (x - 16)(x - 14)(x + 14)(x + 16)$, find the minimum value of y.

(A) -896
(B) -897
(C) -898
(D) -899
(E) -900

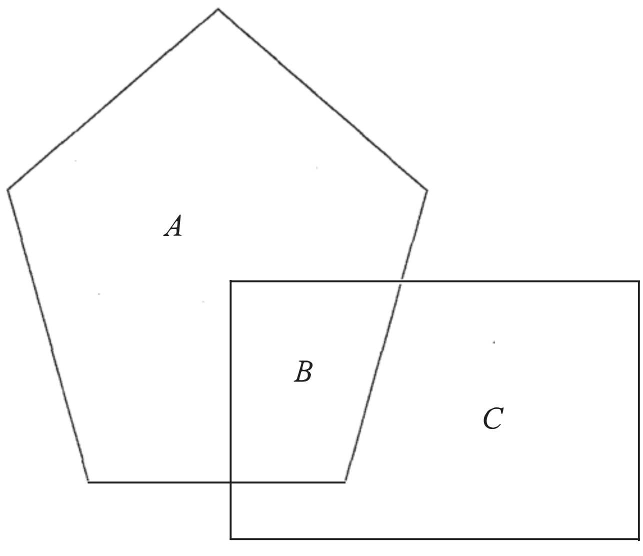
10 The number of positive integral solutions (a, b, c, d) satisfying $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1$ with the condition that $a<b<c<d$ is

(A) 6
(B) 7
(C) 8
(D) 9
(E) 10

Short Questions

11 There are two models of LCD television on sale. One is a '20 inch' standard model while the other is a '20 inch' widescreen model. The ratio of the length to the height of the standard model is 4 : 3, while that of the widescreen model is 16 : 9. Television screens are measured by the length of their diagonals, so both models have the same diagonal length of 20 inches. If the ratio of the area of the standard model to that of the widescreen model is A : 300, find the value of A.

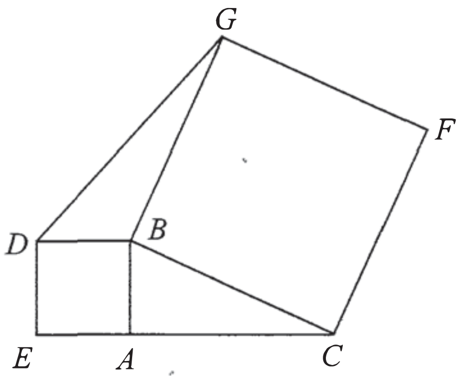
12 The diagram below shows a pentagon (made up of region A and region B) and a rectangle (made up of region B and region C) that overlaps. The overlapped region B is $\frac{3}{16}$ of the pentagon and $\frac{2}{9}$ of the rectangle. If the ratio of region A of the pentagon to region C of the rectangle is $\frac{m}{n}$ in its lowest term, find the value of $m+n$.



13 2009 students are taking a test which comprises ten true or false questions. Find the minimum number of answer scripts required to guarantee two scripts with at least nine identical answers.

14 The number of ways to arrange 5 boys and 6 girls in a row such that girls can be adjacent to other girls but boys cannot be adjacent to other boys is $6! \times k$. Find the value of k.

15 ABC is a right-angled triangle with $\angle BAC = 90°$. A square is constructed on the side AB and BC as shown. The area of the square $ABDE$ is 8 cm^2 and the area of the square $BCFG$ is 26 cm^2. Find the area of triangle DBG in cm^2.



16 The sum of $\frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \frac{1}{4 \times 5 \times 6} + \cdots + \frac{1}{13 \times 14 \times 15} + \frac{1}{14 \times 15 \times 16}$ is $\frac{m}{n}$ in its lowest terms. Find the value of $m + n$.

17 Given that $a + \frac{1}{a+1} = b + \frac{1}{b-1} - 2$ and $a - b + 2 \neq 0$, find the value of $ab - a + b$.

18 If $|x| + x + 5y = 2$ and $|y| - y + x = 7$, find the value of $x + y + 2009$.

19 Let p and q represent two consecutive prime numbers. For some fixed integer n, the set $\{n - 1, 3n - 19, 38 - 5n, 7n - 45\}$ represents $\{p, 2p, q, 2q\}$, but not necessarily in that order. Find the value of n.

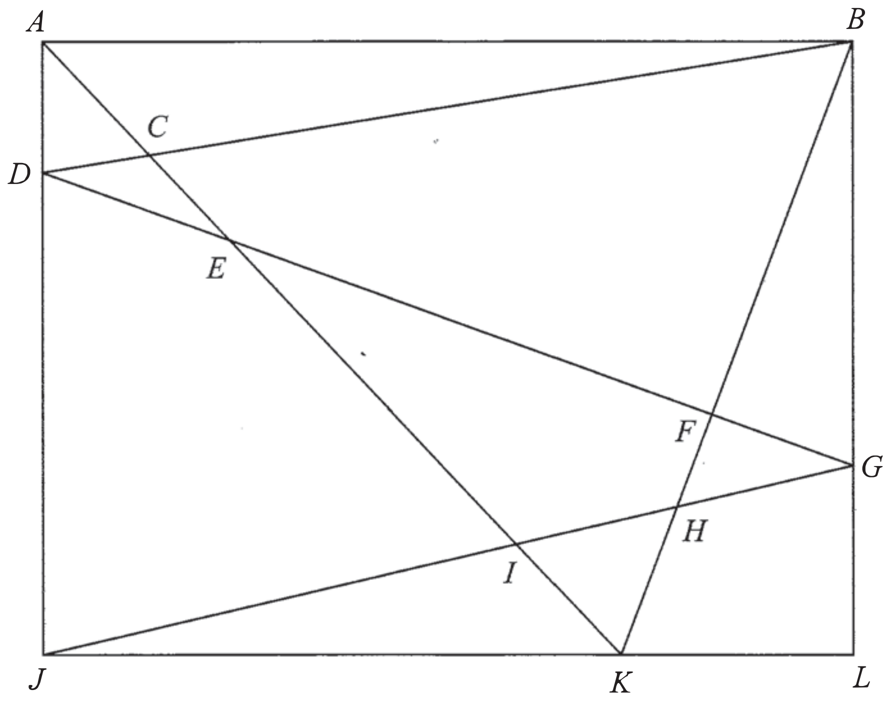
20 Find the number of ordered pairs of positive integers (x, y) that satisfy the equation

$$x\sqrt{y} + y\sqrt{x} + \sqrt{2009xy} - \sqrt{2009x} - \sqrt{2009y} - 2009 = 0.$$

21 Find the integer part of

$$\frac{1}{\frac{1}{2003}+\frac{1}{2004}+\frac{1}{2005}+\frac{1}{2006}+\frac{1}{2007}+\frac{1}{2008}+\frac{1}{2009}}.$$

22 The diagram below shows a rectangle $ABLJ$, where the area of ACD, $BCEF$, $DEIJ$ and FGH are 22 cm^2, 500 cm^2, 482 cm^2 and 22 cm^2 respectively. Find the area of HIK in cm^2.



23 Evaluate $\sqrt[3]{77-20\sqrt{13}} + \sqrt[3]{77+20\sqrt{13}}$.

24 Find the number of integers in the set $\{1, 2, 3, \ldots, 2009\}$ whose sum of the digits is 11.

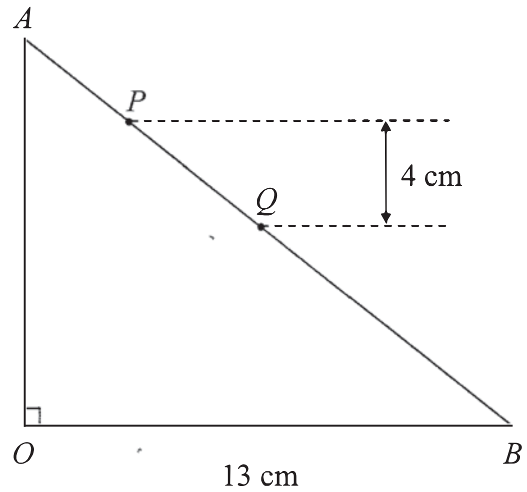
25 Given that

$$x + (1+x)^2 + (1+x)^3 + \ldots + (1+x)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

where each a_r is an integer, $r = 0, 1, 2, \ldots, n$.

Find the value of n such that $a_0 + a_2 + a_3 + a_4 + \ldots + a_{n-2} + a_{n-1} = 60 - \dfrac{n(n+1)}{2}$.

26 In the diagram, OAB is a triangle with $\angle AOB = 90°$ and $OB = 13$ cm. P & Q are 2 points on AB such that $26AP = 22PQ = 11QB$. If the vertical height of $PQ = 4$ cm, find the area of the triangle OPQ in cm^2.



27 Let x_1, x_2, x_3, x_4 denote the four roots of the equation

$$x^4 + kx^2 + 90x - 2009 = 0.$$

If $x_1 x_2 = 49$, find the value of k.

28 Three sides OAB, OAC and OBC of a tetrahedron $OABC$ are right-angled triangles, i.e. $\angle AOB = \angle AOC = \angle BOC = 90°$. Given that $OA = 7$, $OB = 2$ and $OC = 6$, find the value of

$$(\text{Area of } \Delta OAB)^2 + (\text{Area of } \Delta OAC)^2 + (\text{Area of } \Delta OBC)^2 + (\text{Area of } \Delta ABC)^2.$$

29 Find the least positive integer n for which $\dfrac{n-10}{9n+11}$ is a non-zero reducible fraction.

30 Find the value of the smallest positive integer m such that the equation

$$x^2 + 2(m+5)x + (100m+9) = 0$$

has only integer solutions.

31 In a triangle ABC, the length of the altitudes AD and BE are 4 and 12 respectively. Find the largest possible integer value for the length of the third altitude CF.

32 A four digit number consists of two distinct pairs of repeated digits (for example 2211, 2626 and 7007). Find the total number of such possible numbers that are divisible by 7 or 101 but not both.

33 m and n are two positive integers satisfying $1 \leq m \leq n \leq 40$. Find the number of pairs of (m, n) such that their product mn is divisible by 33.

34 Using the digits 0, 1, 2, 3 and 4, find the number of 13-digit sequences that can be written so that the difference between any two consecutive digits is 1.

Examples of such 13-digit sequences are 0123432123432, 2323432321234 and 3210101234323.

35 m and n are two positive integers of reverse order (for example 123 and 321) such that $mn = 1446921630$. Find the value of $m + n$.

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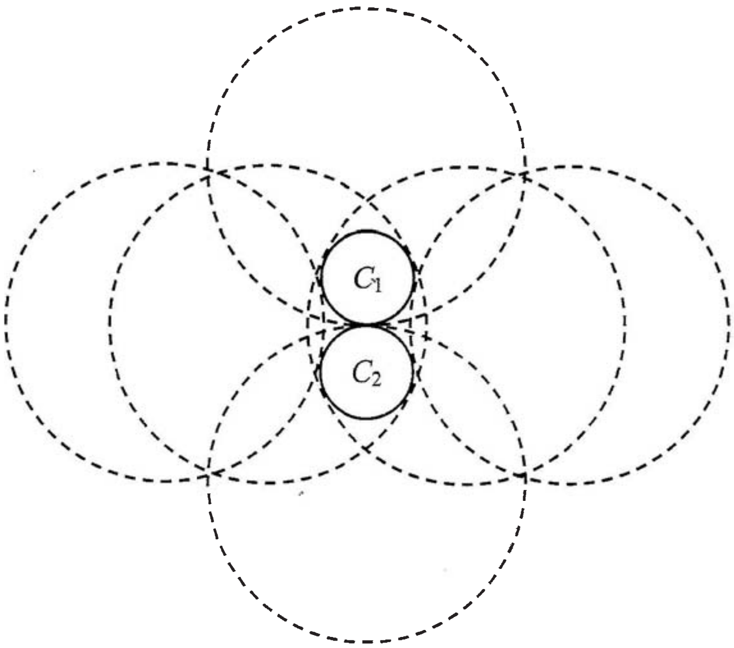
Singapore Mathematical Olympiad (SMO) 2009

(Junior Section Solutions)

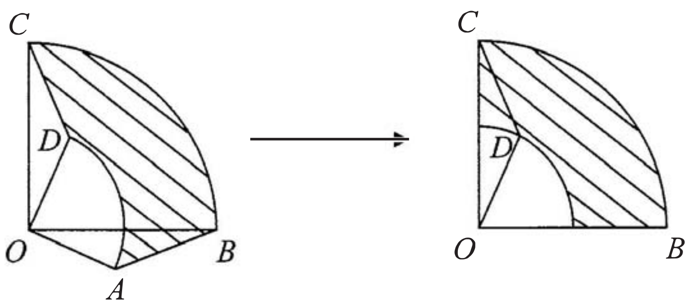
1 Answer: (C)

As seen from the diagram below, there are 6 circles that are tangent to both C_1 and C_2.



2 Answer: (A)

Rotate ΔOAB 90° anticlockwise about the point O to overlap with ΔODC.



$\therefore$ Shaded area $= \frac{1}{4}(64\pi - 16\pi) = 12\pi$.

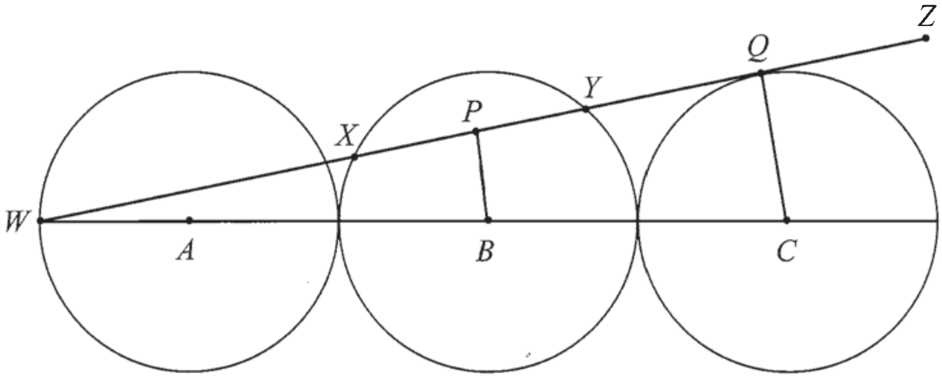
3 Answer: (D)

$$\left(\sqrt{x-2}+\sqrt{7-x}\right)^2 = 5+2\sqrt{(x-2)(7-x)} = 5+2\sqrt{6.25-(x-4.5)^2}.$$

Hence maximum value of $k=\sqrt{5+2(2.5)}=\sqrt{10}$.

4 Answer: (B)

Let P be the midpoint of XY and Q be the point where the line WZ meets the third circle.



Then by similar Δs, $\frac{PB}{60}=\frac{20}{100} \Rightarrow PB = 12$. Hence $XY = 2XP = 2\sqrt{20^2-12^2} = 32$.

5 Answer: (D)

$y=\frac{10x}{10-x} \Rightarrow x = 10-\frac{100}{y+10} < 0$, so $\frac{100}{y+10} > 10 \Rightarrow -10 < y < 0$.

Since both x and y are integers, $y + 10$ is a factor of 100, i.e. $y = -9, -8, -6, -5$.

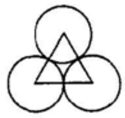
The maximum value of y is -5.

6 Answer: (C)

Let $b_n = a_n - a_{n-1} = n^2$. $\sum_{n=1}^{50} b_n = \sum_{n=1}^{50} n^2 = \frac{20}{100}(50)(51)(101) = 42925$.

On the other hand, $\sum_{n=1}^{50} b_n = a_{50} - a_0 \Rightarrow a_{50} = 42925 + a_0 = 44934$.

7 Answer: (B)



Consider the equilateral triangle formed by joining the centres of 3 adjacent coins. It is easy to see that the required percentage is given by the percentage of the triangle covered by these three coins.

We have $\dfrac{\frac{1}{2}\pi r^2}{\sqrt{3}\,r^2} \times 100\% = \dfrac{50}{\sqrt{3}}\,\pi\,\%$.

8 Answer: (E)

$x^2 + y^2 = 6 \Rightarrow (x+y)^2 = 2xy + 6$. So $2x + \{\,(x+y)^2 - 6\,\} + 2y = 2(2 + 3\sqrt{2})$
$\Rightarrow\ (x+y)^2 + 2(x+y) + 1 = 11 + 6\sqrt{2} \Rightarrow (x+y+1)^2 = (3+\sqrt{2})^2$.

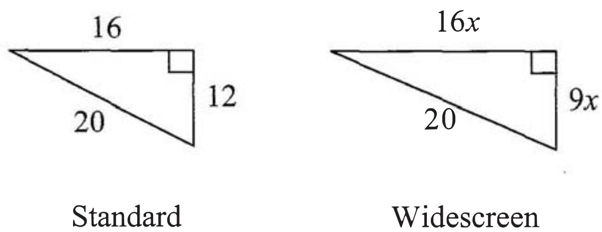
9 Answer: (E)

$y = (x^2 - 16^2)(x^2 - 14^2) = x^4 - 452x^2 + 50176 = (x^2 - 226)^2 - 900$.
So minimum value of $y = -900$.

10 Answer: (A)

$(a, b, c, d) = (2,3,7,42), (2,3,8,24), (2,3,9,18), (2,3,10,15), (2,4,5,20), (2,4,6,12)$.

11 Answer: (337)



$(16x)^2 + (9x)^2 = 20^2 \Rightarrow 337x^2 = 400$. $\therefore$ $\dfrac{\text{Area of Standard}}{\text{Area of Widescreen}} = \dfrac{(16)(12)}{(16x)(9x)} = \dfrac{337}{300}$.

12 Answer: (47)

Let the area of the pentagon and the rectangle be P and R respectively.
We have $\frac{3}{16}P = \frac{2}{9}R$. So $\frac{m}{n} = \frac{13}{16}P \div \frac{7}{9}R = \frac{26}{21} \Rightarrow m + n = 47$.

13 Answer: (513)

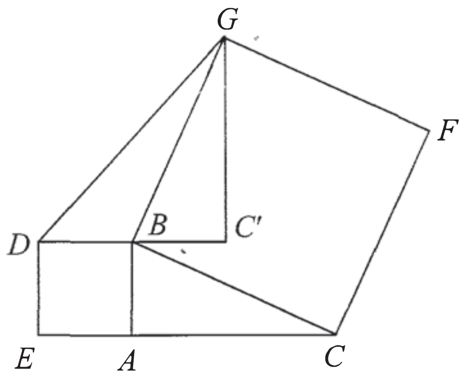
Minimum number of answer scripts is $2^9 + 1 = 513$.

14 Answer: (2520)

First we arrange the 6 girls in 6! ways. Next, there are 7 spaces between the 6 girls to insert the 5 boys. Hence $k = {}^7C_5 \times 5! = 2520$.

15 Answer: (6)

We rotate ΔBAC 90° anticlockwise about the point B to get $\Delta BC'G$.
$C'G = BC = \sqrt{26 - 8} = \sqrt{18}$.



$\therefore$ The area of $\Delta DBG = \frac{1}{2} \times DB \times C'G = \frac{1}{2}\sqrt{8}\sqrt{18} = 6\ \text{cm}^2$.

16 Answer: (173)

Note that $\dfrac{1}{n \times (n+1) \times (n+2)} = \dfrac{1}{2}\left(\dfrac{1}{n \times (n+1)} - \dfrac{1}{(n+1) \times (n+2)}\right)$.
Hence $\dfrac{1}{2 \times 3 \times 4} + \dfrac{1}{3 \times 4 \times 5} + \dfrac{1}{4 \times 5 \times 6} + \dots + \dfrac{1}{13 \times 14 \times 15} + \dfrac{1}{14 \times 15 \times 16}$
$= \dfrac{1}{2}\left(\dfrac{1}{2 \times 3} - \dfrac{1}{15 \times 16}\right) = \dfrac{13}{160}$.

17 Answer: (2)

Let $x = a + 1$, $y = b - 1$ $(x - y \neq 0)$, then $x - 1 + \frac{1}{x} = y + 1 + \frac{1}{y} - 2 \Rightarrow x + \frac{1}{x} = y + \frac{1}{y}$

$\Rightarrow (x - y)\left(1 - \frac{1}{xy}\right) = 0 \Rightarrow xy = 1 \Rightarrow ab - a + b = 2.$

18 Answer: (2012)

Suppose $y \geq 0$. From $|y| - y + x = 7$ we have $x = 7$ and from $|x| + x + 5y = 2$ we have $y = -\frac{12}{5} < 0$ $(\rightarrow\leftarrow)$. So $y < 0$.

Suppose $x \leq 0$. From $|x| + x + 5y = 2$ we have $y = \frac{2}{5} > 0$ $(\rightarrow\leftarrow)$. So $x > 0$.

Hence the two equations become $-2y + x = 7$ and $2x + 5y = 2 \Rightarrow x = \frac{13}{3}$, $y = -\frac{4}{3}$.

19 Answer: (7)

Since $3p + 3q = 6n - 27$, $p + q = 2n - 9$ which is odd. So $p = 2$, $q = 3$ and $n = 7$.

20 Answer: (6)

Note that $x\sqrt{y} + y\sqrt{x} + \sqrt{2009xy} - \sqrt{2009x} - \sqrt{2009y} - 2009$
$= (\sqrt{x} + \sqrt{y} - \sqrt{2009})(\sqrt{xy} - \sqrt{2009})$, so $xy = 2009$.
$\therefore$ $(x, y) = (1, 2009), (7, 287), (41, 49), (49, 41), (287, 7), (2009, 1)$.

21 Answer: (286)

Let $L = \frac{1}{2003} + \frac{1}{2004} + \frac{1}{2005} + \frac{1}{2006} + \frac{1}{2007} + \frac{1}{2008} + \frac{1}{2009}$.

Clearly $\frac{7}{2009} < L < \frac{7}{2003} \Rightarrow 286\frac{1}{7} < \frac{1}{L} < 287$.

22 Answer: (26)

Looking at half of the area of rectangle $ABLJ$, we have $(ABK) = (ABD) + (DGJ)$
$\Rightarrow (ABC) + (BCEF) + (EFHI) + (HIK) = (ABC) + (ACD) + (DEIJ) + (EFHI) + (FGH)$
$\Rightarrow (BCEF) + (HIK) = (ACD) + (DEIJ) + (FGH)$
$\Rightarrow 500 + (HIK) = 22 + 482 + 22 \Rightarrow (HIK) = 26.$

23 Answer: (7)

Let $X = \sqrt[3]{77 - 20\sqrt{13}}$, $Y = \sqrt[3]{77 + 20\sqrt{13}}$ and $A = X + Y$.
Since $X^3 + Y^3 = 154, XY = \sqrt[3]{77^2 - 20^2 \times 13} = 9$, $A^3 = (X + Y)^3 = X^3 + Y^3 + 3XY(X + Y)$
$\Rightarrow A^3 = 154 + 27A \Rightarrow A^3 - 27A - 154 = 0 \Rightarrow (A - 7)(A^2 + 7A + 22) = 0 \Rightarrow A = 7$.

24 Answer: (133)

We consider 3 sets of integers:

(a) 0001 ~ 0999. We count the number of nonnegative solutions of $a + b + c = 11$. It is ${}^{11+3-1}C_{3-1} = {}^{13}C_2 = 78$. Note that 11 should not be split as 0+0+11 nor 0+1+10. So the number of solutions is $78 - 3 - 6 = 69$.

(b) 1001 ~ 1999. We count the number of nonnegative solutions of $a + b + c = 10$. It is ${}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$. Note that 10 should not be split as 0+0+10. So the number of solutions is $66 - 3 = 63$.

(c) 2001 ~ 2009. We see that only 2009 satisfies the property.

$\therefore$ The total number of integers satisfying the property is $69 + 63 + 1 = 133$.

25 Answer: (5)

Let $x = 1$, we have $a_0 + a_1 + a_2 + \dots + a_n = 1 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 3$.
Also, $a_1 = 1 + 2 + 3 + \dots + n = \dfrac{n(n+1)}{2}$, $a_n = 1$,
so $60 - \dfrac{n(n+1)}{2} + \dfrac{n(n+1)}{2} + 1 = 2^{n+1} - 3 \Rightarrow n = 5$.

26 Answer: (26)

Note that area of $\Delta OPQ = \dfrac{1}{2} \times 4 \text{ cm} \times 13 \text{ cm} = 26 \text{ cm}^2$.

27 Answer: (7)

From given equation, $x_1 + x_2 + x_3 + x_4 = 0$, $= x_1 x_2 (x_3 + x_4) + x_3 x_4 (x_1 + x_2) = -90$ and $x_1 x_2 x_3 x_4 = -2009$. Now $x_1 x_2 = 49 \Rightarrow x_3 x_4 = -41$. So $x_1 + x_2 = 1$, $x_3 + x_4 = -1$. Hence $k = x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4 = 49 + (1) \times (-1) - 41 = 7$.

28 Answer: (1052)

Note that (Area of $\Delta OAB)^2 =$ (Area of $\Delta OAC)^2 +$ (Area of $\Delta OBC)^2 +$ (Area of $\Delta ABC)^2$.
So (Area of $\Delta OAB)^2$ + (Area of $\Delta OAC)^2$ + (Area of $\Delta OBC)^2$ + (Area of $\Delta ABC)^2$
$= 2 \times \{$(Area of $\Delta OAC)^2$ + (Area of $\Delta OBC)^2$ + (Area of $\Delta ABC)^2\}$
$= 2 \times \{ (1/2 \times 7 \times 6)^2 + (1/2 \times 7 \times 2)^2 + (1/2 \times 2 \times 6)^2 \} = 1052$.

29 Answer: (111)

Consider $\dfrac{9n+11}{n-10} = 9 + \dfrac{101}{n-10}$. If $\dfrac{n-10}{9n+11}$ is a non-zero reducible fraction, then $\dfrac{101}{n-10}$ is also a non-zero reducible fraction $\Rightarrow$ Least positive integer $n = 111$.

30 Answer: (90)

$x^2 + 2(m+5)x + (100m+9) = 0 \Rightarrow x = -(m+5) \pm \sqrt{(m-45)^2 - 2009}$. This yields integer solutions if and only if $(m-45)^2 - 2009$ is a perfect square, say n^2.

Hence $(m-45)^2 - n^2 = 2009 = 7^2 \times 41 \Rightarrow$

$$|m-45| + n = 2009 \text{ and } |m-45| - n = 1, \text{ or}$$
$$|m-45| + n = 287 \text{ and } |m-45| - n = 7, \text{ or}$$
$$|m-45| + n = 49 \text{ and } |m-45| - n = 41.$$

Solving, $n = 45 \pm 1005, 45 \pm 147, 45 \pm 45 \Rightarrow$ The smallest positive n is 90.

31 Answer: (5)

Area of the triangle is $\frac{1}{2} \times AD \times BC = \frac{1}{2} \times AC \times BE = \frac{1}{2} \times AB \times CF$.

Since $AD = 4$ and $BE = 12$, $BC : AC = 3 : 1$. Let $AC = x$, then $BC = 3x$.

Using Triangle Inequality, $AB < AC + BC$ and $BC < AB + AC \Rightarrow 2x < AB < 4x$.

From $CF = \dfrac{12x}{AB}$, $3 < CF < 6 \Rightarrow$ The largest integer value for CF is 5.

32 Answer: (97)

We let the 2 distinct digits be A and B with $1 \leq A \leq 9$, $0 \leq B \leq 9$ and $A \neq B$.

Case 1: $ABAB = 101 \times AB$. There are $9 \times 9 = 81$ possibilities.
However AB must not be a multiple of 7 (12 possibilities excluding 07 and 77) $\Rightarrow$
There are $81 - 12 = 69$.

Case 2: $AABB = 11 \times (100A + B)$. There are 11 possibilities that are factors of 7.
$(A,B) = (1,5)$, $(2,3)$, $(3,1)$, $(3,8)$, $(4,6)$, $(5,4)$, $(6,2)$, $(6,9)$, $(7,0)$, $(8,5)$, $(9,3)$.

Case 3: $ABBA = 11 \times (91A + 10B)$. B must be 0 or 7. There are 17 possibilities.

$\therefore$ Total such possible number $= 69 + 11 + 17 = 97$.

33 Answer: (64)

If $n = 33$, $m = 1, 2, 3, \ldots, 32$, so there are 32 pairs.
If $m = 33$, $n = 33, 34, \ldots, 40$, so there are 8 pairs.

If $n, m \neq 33$, we have 2 cases:

Case 1: $m = 3a$, $n = 11b$ ($a \neq 11$ and $b \neq 3$), thus $1 \leq 3a \leq 11b \leq 40$.
So if $b = 1$, $a = 1, 2, 3$ and if $b = 2$, $a = 1, 2, 3, \ldots, 7$. There are 10 pairs.

Case 2: $m = 11a$, $n = 3b$ ($a \neq 3$ and $b \neq 11$), thus $1 \leq 11a \leq 3b \leq 40$.
So if $a = 1$, $b = 4, 5, 6, 7, 8, 9, 10, 12, 13$ and if $a = 2$, $b = 8, 9, 10, 12, 13$.
There are $9 + 5 = 14$ pairs.

Hence we have altogether $32 + 8 + 10 + 14 = 64$ pairs.

34 Answer: (3402)

We call a sequence that satisfy the condition a "good" sequence.

Let A_n denote the number of "good" sequence that end in either 0 or 4,
B_n denote the number of "good" sequence that end in either 1 or 3,
C_n denote the number of "good" sequence that end in 2.

We have

(1) $A_{n+1} = B_n$ because each sequence in A_{n+1} can be converted to a sequence in B_n by deleting its last digit.

(2) $B_{n+1} = A_n + 2C_n$ because each sequence in A_n can be converted into a sequence in B_{n+1} by adding a 1 (if it ends with a 0) or a 3 (if it ends with a 4) to its end, and each sequence in C_n can be converted into a sequence in B_n by adding a 1 or a 3 to it.

(3) $C_{n+1} = B_n$ because each sequence in C_{n+1} can be converted to a sequence in B_n by deleting the 2 at its end.

Hence we can show that $B_{n+1} = 3B_{n-1}$ for $n \geq 2$.

Check that $B_1 = 2$ and $B_2 = 4$, so $B_{2n+1} = 2 \times 3^n$ and $B_{2n} = 4 \times 3^{n-1}$.

So $A_{13} + B_{13} + C_{13} = 2B_{12} + B_{13} = 2 \times 4 \times 3^5 + 2 \times 3^6 = 3402$.

35 Answer: (79497)

Clearly, m and n are both 5-digit numbers.

Next, it would be helpful that we know $mn = 2 \times 3^5 \times 5 \times 7 \times 11^2 \times 19 \times 37$.

Now since the last digit of mn is 0, we may assume $5 \mid m$ and $2 \mid n$. But the first digit of mn is 1 $\Rightarrow$ Last digit of m is 5 (not 0) and last digit of n is 2 (not 4, 6 or 8).

Also, $3^5 \mid mn$, so 9 divides at least one of m and n. On the other hand, $9 \mid m \Rightarrow 9 \mid n$. Similarly $11 \mid m \Rightarrow 11 \mid n$.

Set $n = 198k$. Then the last digit of k is 4 or 9.

Since the remaining factors 3, 7, 19, 37 are odd, the last digit of k must be 9.
We have only the following combinations: $k = 7 \times 37$ or $3 \times 7 \times 19$ or $3 \times 19 \times 37$.

Recall that the first digit of n is 5, so $50000 \leq 198k < 60000 \Rightarrow k = 7 \times 37$.

Hence $n = 198k = 51282$ and $m = 28215 \Rightarrow m + n = 79497$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Junior Section, Round 2)

1. In $\triangle ABC$, $\angle A = 2\angle B$. Let a, b, c be the lengths of its sides BC, CA, AB, respectively. Prove that

$$a^2 = b(b + c).$$

2. The set of 2000-digit integers are divided into two sets: the set M consisting all integers each of which can be represented as the product of two 1000-digit integers, and the set N which contains the other integers. Which of the sets M and N contains more elements?

3. Suppose $\overline{a_1 a_2 \ldots a_{2009}}$ is a 2009-digit integer such that for each $i = 1, 2, \ldots, 2007$, the 2-digit integer $\overline{a_i a_{i+1}}$ contains 3 distinct prime factors. Find a_{2008}. (Note: $\overline{xyz\ldots}$ denotes an integer whose digits are $x, y, z, \ldots$.)

4. Let S be the set of integers that can be written in the form $50m + 3n$ where m and n are non-negative integers. For example 3, 50, 53 are all in S. Find the sum of all positive integers not in S.

5. Let a, b be positive real numbers satisfying $a + b = 1$. Show that if $x_1, x_2, \ldots, x_5$ are positive real numbers such that $x_1 x_2 \ldots x_5 = 1$, then

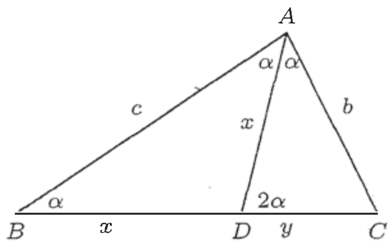
$$(ax_1 + b)(ax_2 + b) \cdots (ax_5 + b) \geq 1.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Junior Section, Round 2 solutions)

1. Let AD be the angle bisector of $\angle A$. Then $\triangle ABC \simeq \triangle DAC$. Thus $AB/DA = AC/DC = BC/AC$. Let $BD = x$ and $DC = y$. Then $c/x = b/y = a/b$. Thus $cb = ax, b^2 = ay$. Thus $b^2 + cb = ax + ay$ and hence $b(b+c) = a^2$.



2. We solve the general case of $2n$-digit integers where $n \geq 2$. There are $10^{2n} - 10^{2n-1}$ $2n$-digit integers. There are $10^n - 10^{n-1}$ n-digit integers. Consider all the products of pairs of n-digit integers. The total number P of such products satisfies

$$\begin{aligned}
P &\leq 10^n - 10^{n-1} + \frac{(10^n - 10^{n-1})(10^n - 10^{n-1} - 1)}{2} \\
&= \frac{10^{2n} - 10^{2n-1} - (10^{2n-1} - 10^{2n-2} - 10^n + 10^{n-1})}{2} \\
&< \frac{10^{2n} - 10^{2n-1}}{2}.
\end{aligned}$$

These products include all the numbers in M. Thus $|M| < |N|$.

3. Two-digit numbers which contain three distinct prime factors are:

$30 = 2\cdot3\cdot5, 42 = 2\cdot3\cdot7, 60 = 4\cdot3\cdot5, 66 = 2\cdot3\cdot11, 70 = 2\cdot5\cdot7, 78 = 2\cdot3\cdot13, 84 = 4\cdot3\cdot7$

From here, we conclude that $a_i = 6$ for $i = 1, 2, \ldots, 2007$ and a_{2008} is either 6 or 0.

4. If x is the smallest integer in S such that $x \equiv i \pmod 3$, then $x + 3k \in S$ and $x - 3(k+1) \notin S$ for all $k \geq 0$. We have 3 is the smallest multiple of 3 that is in S; 50 is smallest number in S that is $\equiv 2 \pmod 3$ and 100 is the smallest number in S that is $\equiv 1 \pmod 3$. Thus the positive numbers not in S are $1, 4, \ldots, 97$ and $2, 5, \ldots, 47$. Their sum is

$$\frac{33(97+1)}{2} + \frac{16(2+47)}{2} = 2009.$$

5. The left hand side is

$$\begin{aligned}
a^5 x_1 x_2 \ldots x_5 &+ a^4 b(x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_2x_3x_4x_5) \\
&+ a^3b^2(x_1x_2x_3 + x_1x_2x_4 + \cdots + x_3x_4x_5) \\
&+ a^2b^3(x_1x_2 + x_1x_3 + \cdots + x_4x_5) + ab^4(x_1 + x_2 + \cdots + x_5) + b^5 \\
&\geq a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = (a+b)^5 = 1.
\end{aligned}$$

The last is true since by AM-GM inequality,

$$\begin{aligned}
&x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_2x_3x_4x_5 \geq 5(x_1x_2x_3x_4x_5)^{4/5} = 5 \\
&x_1x_2x_3 + x_1x_2x_4 + \cdots + x_3x_4x_5 \geq 10(x_1x_2x_3x_4x_5)^{6/10} = 10 \\
&x_1x_2 + x_1x_3 + \cdots + x_4x_5 \geq 10(x_1x_2x_3x_4x_5)^{4/10} = 10 \\
&x_1 + x_2 + \cdots + x_5 \geq 5(x_1x_2x_3x_4x_5)^{1/5} = 5
\end{aligned}$$

(Note. For a proof of the general case, see Senior Q4)

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section)

Tuesday, 2 June 2009 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D *or* E) *corresponding to the correct answer.*

For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Suppose that π is a plane and A and B are two points on the plane π. If the distance between A and B is 33 cm, how many lines are there in the plane such that the distance between each line and A is 7 cm and the distance between each line and B is 26 cm respectively?

   (A) 1
   (B) 2
   (C) 3
   (D) 4
   (E) Infinitely many

2. Let $y = (17 - x)(19 - x)(19 + x)(17 + x)$, where x is a real number. Find the smallest possible value of y.

   (A) -1296
   (B) -1295
   (C) -1294
   (D) -1293
   (E) -1292

3. If two real numbers a and b are randomly chosen from the interval (0, 1), find the probability that the equation $x^2 - \sqrt{a}\,x + b = 0$ has real roots.

   (A) $\frac{1}{8}$
   (B) $\frac{5}{16}$
   (C) $\frac{3}{16}$
   (D) $\frac{1}{4}$
   (E) $\frac{1}{3}$

4. If x and y are real numbers for which $|x| + x + 5y = 2$ and $|y| - y + x = 7$, find the value of $x + y$.

   (A) -3
   (B) -1
   (C) 1
   (D) 3
   (E) 5

5. In a triangle ABC, $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$. Find the value of $\cos C$.

(A) $\frac{56}{65}$ or $\frac{16}{65}$

(B) $\frac{56}{65}$

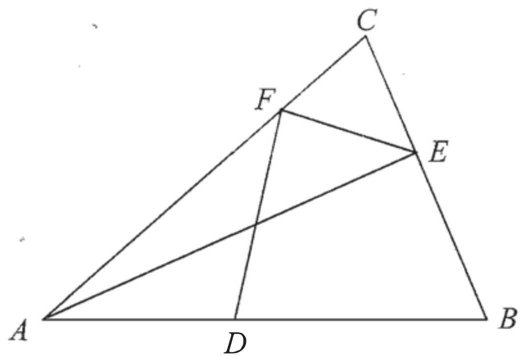
(C) $\frac{16}{65}$

(D) $-\frac{56}{65}$

(E) $\frac{56}{65}$ or $-\frac{16}{65}$

6. The area of a triangle ABC is 40 cm^2. Points D, E and F are on sides AB, BC and CA respectively, as shown in the figure below. If $AD = 3$ cm, $DB = 5$ cm, and the area of triangle ABE is equal to the area of quadrilateral $DBEF$, find the area of triangle AEC in cm^2.

(A) 11
(B) 12
(C) 13
(D) 14
(E) 15



7. Find the value of $\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\cdots+\frac{22}{20!+21!+22!}$.

(A) $1-\frac{1}{24!}$

(B) $\frac{1}{2}-\frac{1}{23!}$

(C) $\frac{1}{2}-\frac{1}{22!}$

(D) $1-\frac{1}{22!}$

(E) $\frac{1}{2}-\frac{1}{24!}$

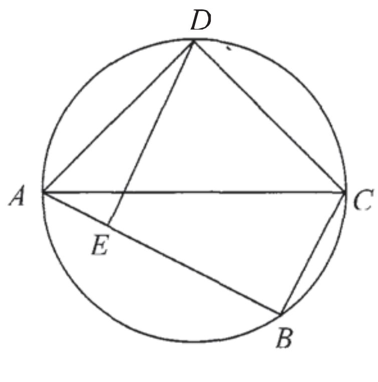
8. There are eight envelopes numbered 1 to 8. Find the number of ways in which 4 identical red buttons and 4 identical blue buttons can be put in the envelopes such that each envelope contains exactly one button, and the sum of the numbers on the envelopes containing the red buttons is more than the sum of the numbers on the envelopes containing the blue buttons.

(A) 35
(B) 34
(C) 32
(D) 31
(E) 62

9. Determine the number of acute-angled triangles (i.e., all angles are less than 90°) in which all angles (in degrees) are positive integers and the largest angle is three times the smallest angle.

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

10. Let $ABCD$ be a quadrilateral inscribed in a circle with diameter AC, and let E be the foot of perpendicular from D onto AB, as shown in the figure below. If $AD = DC$ and the area of quadrilateral $ABCD$ is 24 cm^2, find the length of DE in cm.



(A) $3\sqrt{2}$
(B) $2\sqrt{6}$
(C) $2\sqrt{7}$
(D) $4\sqrt{2}$
(E) 6

Short Questions

11. Find the number of positive divisors of $(2008^3 + (3 \times 2008 \times 2009) + 1)^2$.

12. Suppose that a, b and c are real numbers greater than 1. Find the value of

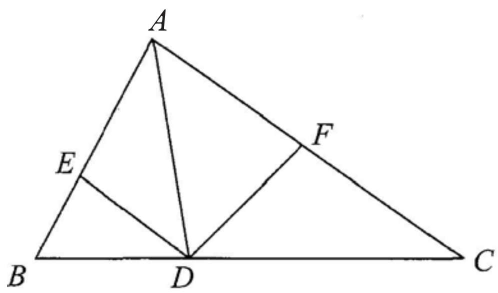
$$\frac{1}{1+\log_{a^2b}\left(\frac{c}{a}\right)}+\frac{1}{1+\log_{b^2c}\left(\frac{a}{b}\right)}+\frac{1}{1+\log_{c^2a}\left(\frac{b}{c}\right)}.$$

13. Find the remainder when $(1! \times 1) + (2! \times 2) + (3! \times 3) + \cdots + (286! \times 286)$ is divided by 2009.

14. Find the value of $\left(25+10\sqrt{5}\right)^{1/3}+\left(25-10\sqrt{5}\right)^{1/3}$.

15. Let $a=\frac{1+\sqrt{2009}}{2}$. Find the value of $(a^3-503a-500)^{10}$.

16. In the figure below, ABC is a triangle and D is a point on side BC. Point E is on side AB such that DE is the angle bisector of $\angle ADB$, and point F is on side AC such that DF is the angle bisector of $\angle ADC$. Find the value of $\frac{AE}{EB}\cdot\frac{BD}{DC}\cdot\frac{CF}{FA}$.



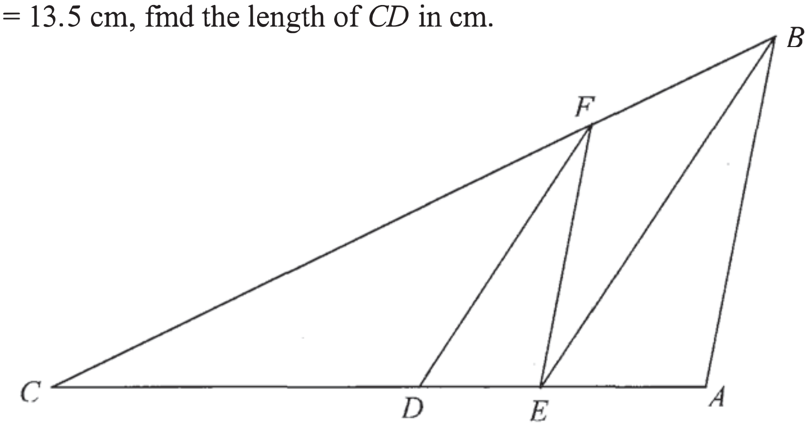
17. Find the value of

$$(\cot 25^\circ-1)(\cot 24^\circ-1)(\cot 23^\circ-1)(\cot 22^\circ-1)(\cot 21^\circ-1)(\cot 20^\circ-1).$$

18. Find the number of 2-element subsets $\{a, b\}$ of $\{1, 2, 3, \ldots, 99, 100\}$ such that $ab + a + b$ is a multiple of 7.

19. Let x be a real number such that $x^2 - 15x + 1 = 0$. Find the value of $x^4 + \frac{1}{x^4}$.

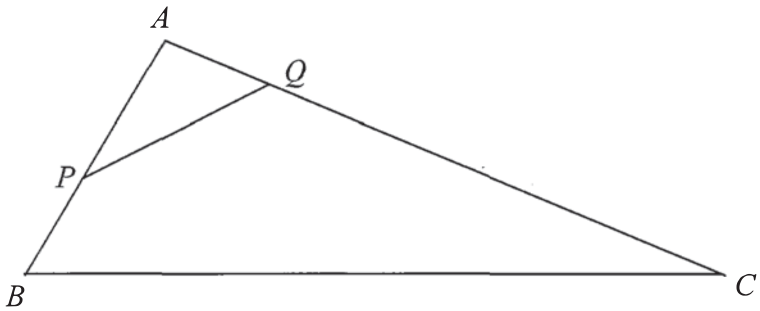
20. In the figure below, ABC is a triangle with $AB = 10$ cm and $BC = 40$ cm. Points D and E lie on side AC and point F on side BC such that EF is parallel to AB and DF is parallel to EB. Given that BE is an angle bisector of $\angle ABC$ and that $AD = 13.5$ cm, find the length of CD in cm.



21. Let $S = \{1, 2, 3, \ldots, 64, 65\}$. Determine the number of ordered triples (x, y, z) such that $x, y, z \in S$, $x < z$ and $y < z$.

22. Given that $a_{n+1} = \dfrac{a_{n-1}}{1 + n a_{n-1} a_n}$, where $n = 1, 2, 3, \ldots$, and $a_0 = a_1 = 1$, find the value of $\dfrac{1}{a_{199}\, a_{200}}$.

23. In the figure below, ABC is a triangle with $AB = 5$ cm, $BC = 13$ cm and $AC = 10$ cm. Points P and Q lie on sides AB and AC respectively such that $\dfrac{\text{area of } \Delta APQ}{\text{area of } \Delta ABC} = \dfrac{1}{4}$. Given that the least possible length of PQ is k cm, find the value of k.



24. If x, y and z are real numbers such that $x + y + z = 9$ and $xy + yz + zx = 24$, find the largest possible value of z.

25. Find the number of 0 – 1 binary sequences formed by six 0's and six 1's such that no three 0's are together. For example, 110010100101 is such a sequence but 101011000101 and 110101100001 are not.

26. If $\dfrac{\cos 100^\circ}{1 - 4\sin 25^\circ \cos 25^\circ \cos 50^\circ} = \tan x^\circ$, find x.

27. Find the number of positive integers x, where $x \neq 9$, such that

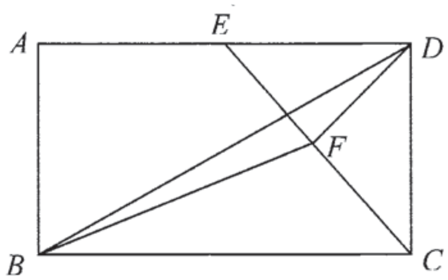
$$\log_{\frac{x}{9}}\left(\frac{x^2}{3}\right) < 6 + \log_3\left(\frac{9}{x}\right).$$

28. Let n be the positive integer such that

$$\frac{1}{9\sqrt{11}+11\sqrt{9}}+\frac{1}{11\sqrt{13}+13\sqrt{11}}+\frac{1}{13\sqrt{15}+15\sqrt{13}}+\cdots+\frac{1}{n\sqrt{n+2}+(n+2)\sqrt{n}}=\frac{1}{9}.$$

Find the value of n.

29. In the figure below, $ABCD$ is a rectangle, E is the midpoint of AD and F is the midpoint of CE. If the area of triangle BDF is 12 cm^2, find the area of rectangle $ABCD$ in cm^2.



30. In each of the following 6-digit positive integers: 555555, 555333, 818811, 300388, every digit in the number appears at least twice. Find the number of such 6-digit positive integers.

31. Let x and y be positive integers such that $27x + 35y \leq 945$. Find the largest possible value of xy.

32. Determine the coefficient of x^{29} in the expansion of $(1+x^5+x^7+x^9)^{16}$.

33. For $n = 1, 2, 3, \ldots$, let $a_n = n^2 + 100$, and let d_n denote the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges over all positive integers.

34. Using the digits 1, 2, 3, 4, 5, 6, 7, 8, we can form $8!$ (= 40320) 8-digit numbers in which the eight digits are all distinct. For $1 \le k \le 40320$, let a_k denote the kth number if these numbers are arranged in increasing order:

$$12345678,\ 12345687,\ 12345768,\ \ldots,\ 87654321;$$

that is, $a_1 = 12345678$, $a_2 = 12345687$, $\ldots$, $a_{40320} = 87654321$. Find $a_{2009} - a_{2008}$.

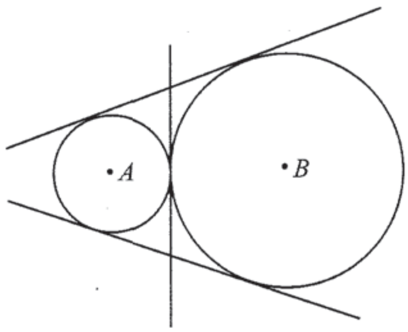
35. Let x be a positive integer, and write $a = \lfloor \log_{10} x \rfloor$ and $b = \left\lfloor \log_{10} \dfrac{100}{x} \right\rfloor$. Here $\lfloor c \rfloor$ denotes the greatest integer less than or equal to c. Find the largest possible value of $2a^2 - 3b^2$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section Solutions)

1. Answer: (C)
In the plane π, draw a circle of radius 7 cm centred at A and a circle of radius 26 cm centred at B.



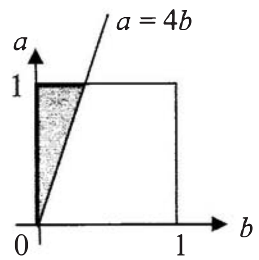
If ℓ is a line on the plane π, and the distance between ℓ and A is 7 cm and the distance between ℓ and B is 26 cm, then ℓ must be tangential to both circles. Clearly, there are 3 lines in the plane that are tangential to both circles, as shown in the figure above.

2. Answer: (A)
We have

$$\begin{aligned} y &= (17^2 - x^2)(19^2 - x^2) \\ &= x^4 - (17^2 + 19^2)x^2 + 17^2 \cdot 19^2 \\ &= x^4 - 650x^2 + 323^2 \\ &= (x^2 - 325)^2 + 323^2 - 325^2 \end{aligned}$$

Hence the smallest possible value of y is $323^2 - 325^2 = (-2)(648) = -1296$.

3. Answer: (A)
The discriminant of the equation is $a - 4b$.
Thus the equation has real roots if and only if $a \geq 4b$.
The shaded part in the figure on the right are all the points with coordinates (a, b) such that $0 < a, b < 1$ and $a \geq 4b$. As the area of the shaded part is $\frac{1}{8}$, it follows that the required probability is $\frac{1}{8}$.



4. Answer: (D)

If $x \le 0$, then $|x| = -x$, and we obtain from $|x| + x + 5y = 2$ that $y = \frac{2}{5}$. Thus y is positive, so $|y| - y + x = 7$ gives $x = 7$, which is a contradiction since $x \le 0$. Therefore we must have $x > 0$. Consequently, $|x| + x + 5y = 2$ gives the equation

$$2x + 5y = 2. \quad (1)$$

If $y \ge 0$, then $|y| - y + x = 7$ gives $x = 7$. Substituting $x = 7$ into $|x| + x + 5y = 2$, we get $y = -\frac{12}{5}$, which contradicts $y \ge 0$. Hence we must have $y < 0$, and it follows from the equation $|y| - y + x = 7$ that

$$x - 2y = 7. \quad (2)$$

Solving equations (1) and (2) gives $x = \frac{13}{3}$, $y = -\frac{4}{3}$. Therefore $x + y = 3$.

5. Answer: (C)

$\sin A = \frac{3}{5}$ implies that $\cos A = \frac{4}{5}$ or $-\frac{4}{5}$, and $\cos B = \frac{5}{13}$ implies that $\sin B = \frac{12}{13}$, since $0 < B < 180^\circ$.

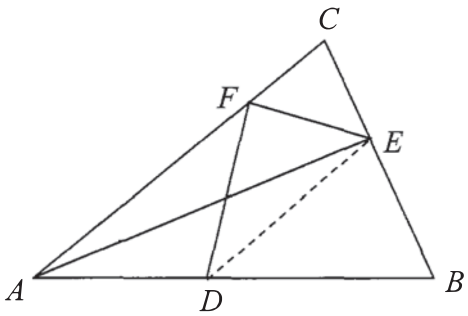
If $\cos A = -\frac{4}{5}$, then $\sin(A+B) = \sin A \cos B + \cos A \sin B = \frac{3}{5}\cdot\frac{5}{13} - \frac{4}{5}\cdot\frac{12}{13} < 0$, which is not possible since $0 < A + B < 180^\circ$ in a triangle. Thus we must have $\cos A = \frac{4}{5}$. Consequently, since $C = 180^\circ - (A + B)$, we have

$$\begin{aligned}\cos C &= -\cos(A+B) = -\cos A \cos B + \sin A \sin B \\ &= -\frac{4}{5}\cdot\frac{5}{13} + \frac{3}{5}\cdot\frac{12}{13} = \frac{16}{65}.\end{aligned}$$

6. Answer: (E)

Since area of triangle ABE is equal to area of quadrilateral $DBEF$, we see that area of ΔDEA = area of ΔDEF. This implies that DE is parallel to AF.

Thus $\frac{CE}{CB} = \frac{AD}{AB} = \frac{3}{8}$. Since $\frac{\text{area of } \Delta AEC}{\text{area of } \Delta ABC} = \frac{CE}{CB}$, it follows that area of $\Delta AEC = \frac{3}{8} \times 40 = 15\ \text{cm}^2$.



7. Answer: (C)

First note that $(n-2)!+(n-1)!+n!=(n-2)![1+(n-1)+n(n-1)]=n^2(n-2)!$.

Therefore the given series can be written as

$$\sum_{n=3}^{22}\frac{n}{n^2(n-2)!}=\sum_{n=3}^{22}\frac{1}{n(n-2)!}=\sum_{n=3}^{22}\frac{n-1}{n(n-1)(n-2)!}$$

$$=\sum_{n=3}^{22}\frac{n-1}{n!}=\sum_{n=3}^{22}\left(\frac{1}{(n-1)!}-\frac{1}{n!}\right).$$

Summing the telescoping series, we obtain $\frac{1}{2!}-\frac{1}{22!}$.

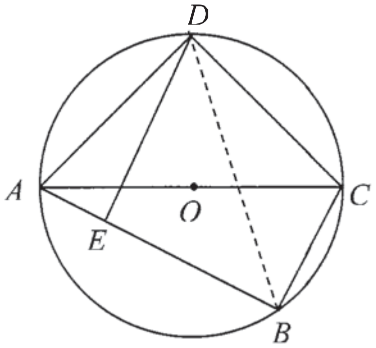
8. Answer: (D)

There are $\binom{8}{4}=70$ ways of putting 4 identical red buttons and 4 identical blue buttons in the envelopes. Since $1+2+3+\cdots+8=36$, there are 8 cases where the sum of the numbers on the envelopes containing the red buttons is equal to 18 (which is also equal to the sum of the numbers on the envelopes containing the blue buttons), namely, (8, 7, 2, 1), (8, 6, 3, 1), (8, 5, 4, 1), (8, 5, 3, 2), (7, 6, 4, 1), (7, 6, 3, 2), (7, 5, 4, 2) and (6, 5, 4, 3). Hence it follows that the required number of ways is $\frac{70-8}{2}=31$.

9. Answer: (B)

Let the angles of the acute-angled triangle be x°, y°, $3x^\circ$, where the smallest angle is x°. Then we have $x+y+3x=180$ and $0<x\leq y\leq 3x<90$. From the inequalities $x\leq y\leq 3x$, we obtain $5x\leq x+y+3x\leq 7x$, and hence it follows from the first equation that $\frac{180}{7}\leq x\leq 36$. Since x is an integer and $3x<90$, we deduce that $x=26, 27, 28, 29$. Hence there are 4 acute-angled triangles whose angles are respectively $(26^\circ, 76^\circ, 78^\circ)$, $(27^\circ, 72^\circ, 81^\circ)$, $(28^\circ, 68^\circ, 84^\circ)$ and $(29^\circ, 64^\circ, 87^\circ)$.

10. Answer: (B)



Let r be the radius of the circle with centre O.
Since $AD = DC$ and $\angle ADC = 90^\circ$, $\angle ACD = 45^\circ$. Thus $\angle ABD = 45^\circ$. As $\angle DEB = 90^\circ$, this implies that $DE = BE$. Let $x = DE = BE$. Since $BC /\!/ ED$ and

area of quadrilateral $ABCD$ = area of ΔAED + area of ΔEBD + area of ΔBCD,

we have

$$24 = \frac{1}{2}\cdot(AB - x)x + \frac{1}{2}x^2 + \frac{1}{2}\cdot BC\cdot x = \frac{1}{2}(AB+BC)x. \quad (1)$$

On the other hand, as OD is perpendicular to AC, and

area of quadrilateral $ABCD$ = area of ΔABC + area of ΔACD,

we have

$$24 = \frac{1}{2}\cdot AB\cdot BC + \frac{1}{2}AC\cdot OD = \frac{1}{2}\cdot AB\cdot BC + r^2. \quad (2)$$

Now equation (2) and $AB^2 + BC^2 = AC^2 = 4r^2$ imply that

$$(AB+BC)^2 = 4r^2 + 2\cdot AB\cdot BC = 4r^2 + 4(24 - r^2) = 96.$$

Therefore $AB + BC = 4\sqrt{6}$. Hence from equation (1), we obtain $x = 2\sqrt{6}$.

11. Answer: 91

Let $a = 2008$. Then

$$\begin{aligned}(2008^3 + (3\times 2008\times 2009) + 1)^2 &= (a^3 + 3a(a+1) + 1)^2\\ &= (a^3 + 3a^2 + 3a + 1)^2\\ &= (a+1)^6 = 2009^6 = 7^{12}\cdot 41^6.\end{aligned}$$

Hence the number of positive divisors is $(12 + 1)(6 + 1) = 91$.

12. Answer: 3

$$\frac{1}{1+\log_{a^2b}\left(\frac{c}{a}\right)}+\frac{1}{1+\log_{b^2c}\left(\frac{a}{b}\right)}+\frac{1}{1+\log_{c^2a}\left(\frac{b}{c}\right)}$$

$$=\frac{1}{\log_{a^2b}(a^2b)+\log_{a^2b}\left(\frac{c}{a}\right)}+\frac{1}{\log_{b^2c}(b^2c)+\log_{b^2c}\left(\frac{a}{b}\right)}+\frac{1}{\log_{c^2a}(c^2a)+\log_{c^2a}\left(\frac{b}{c}\right)}$$

$$=\frac{1}{\log_{a^2b}(abc)}+\frac{1}{\log_{b^2c}(abc)}+\frac{1}{\log_{c^2a}(abc)}$$

$$=\log_{abc}(a^2b)+\log_{abc}(b^2c)+\log_{abc}(c^2a)$$

$$=\log_{abc}(abc)^3=3.$$

13. Answer: 2008

Observe that $n! \times n = n! \times (n+1-1) = (n+1)! - n!$. Therefore

$$(1!\times 1)+(2!\times 2)+(3!\times 3)+\cdots+(286!\times 286)$$
$$=(2!-1!)+(3!-2!)+(4!-3!)+\cdots+(287!-286!)$$
$$=287!-1.$$

Since $2009 = 287 \times 7$, $287! - 1 \equiv -1 \equiv 2008 \pmod{2009}$. It follows that the remainder is 2008.

14. Answer: 5

Let $x=\left(25+10\sqrt{5}\right)^{1/3}+\left(25-10\sqrt{5}\right)^{1/3}$. Then

$$x^3=\left(25+10\sqrt{5}+25-10\sqrt{5}\right)+3\left(25^2-100(5)\right)^{1/3}\left[\left(25+10\sqrt{5}\right)^{1/3}+\left(25-10\sqrt{5}\right)^{1/3}\right],$$

which gives $x^3=50+15x$, or $(x-5)(x^2+5x+10)=0$. This equation admits only one real root $x = 5$.

15. Answer: 1024

$a=\dfrac{1+\sqrt{2009}}{2}$ gives $(2a-1)^2=2009$, which simplified to $a^2-a=502$. Now

$$\left(a^3-503a-500\right)^{10}=\left(a(a^2-a-502)+a^2-a-500\right)^{10}$$
$$=\left(a(a^2-a-502)+(a^2-a-502)+2\right)^{10}$$
$$=(0+0+2)^{10}=1024.$$

16. Answer: 1

Since DE is the angle bisector of $\angle ADB$, we have $\dfrac{AE}{EB}=\dfrac{AD}{BD}$. Similarly, since DF is the angle bisector of $\angle ADC$, $\dfrac{AF}{CF}=\dfrac{AD}{DC}$. Hence $\dfrac{AE}{EB}\cdot\dfrac{BD}{DC}\cdot\dfrac{CF}{FA}=1$.

17. Answer: 8

First note that if $A+B=45^\circ$, then $1=\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$, and so $1-\tan A-\tan B=\tan A\tan B$. Consequently,

$$(\cot A-1)(\cot B-1)=\frac{1-\tan A-\tan B+\tan A\tan B}{\tan A\tan B}=\frac{2\tan A\tan B}{\tan A\tan B}=2.$$

Hence

$$\begin{aligned}&(\cot 25^\circ-1)(\cot 24^\circ-1)(\cot 23^\circ-1)(\cot 22^\circ-1)(\cot 21^\circ-1)(\cot 20^\circ-1)\\ &=(\cot 25^\circ-1)(\cot 20^\circ-1)(\cot 24^\circ-1)(\cot 21^\circ-1)(\cot 23^\circ-1)(\cot 22^\circ-1)\\ &=8.\end{aligned}$$

18. Answer: 602

Note that $ab+a+b=(a+1)(b+1)-1$. Thus $ab+a+b$ is a multiple of 7 if and only if $(a+1)(b+1)\equiv 1 \pmod 7$.

Let $A=\{1, 2, 3, \ldots, 99, 100\}$, and let $A_i=\{x\in A : x\equiv i \pmod 7\}$ for $i=0, 1, 2, \ldots, 6$. It is easy to verify that for any $x\in A_i$ and $y\in A_j$, where $0\le i\le j\le 6$, $xy\equiv 1 \pmod 7$ if and only if $i=j\in\{1, 6\}$, or $i=2$ and $j=4$, or $i=3$ and $j=5$. Thus we consider three cases.

Case 1: $a+1, b+1\in A_i$ for $i\in\{1, 6\}$.

Then $a, b\in A_i$ for $i\in\{0, 5\}$. As $|A_0|=|A_5|=14$, the number of such subsets $\{a, b\}$ is $2\dbinom{14}{2}=182$.

Case 2: $a+1$ and $b+1$ are contained in A_2 and A_4 respectively, but not in the same set.

Then a and b are contained in A_1 and A_3 respectively, but not in the same set. Since $|A_1|=15$ and $|A_3|=14$, the number of such subsets $\{a, b\}$ is $14\times 15=210$.

Case 3: $a+1$ and $b+1$ are contained in A_3 and A_5 respectively, but not in the same set.

Then a and b are contained in A_2 and A_4 respectively, but not in the same set. Note that $|A_2|=15$ and $|A_4|=14$. Thus the number of such subsets $\{a, b\}$ is $14\times 15=210$.

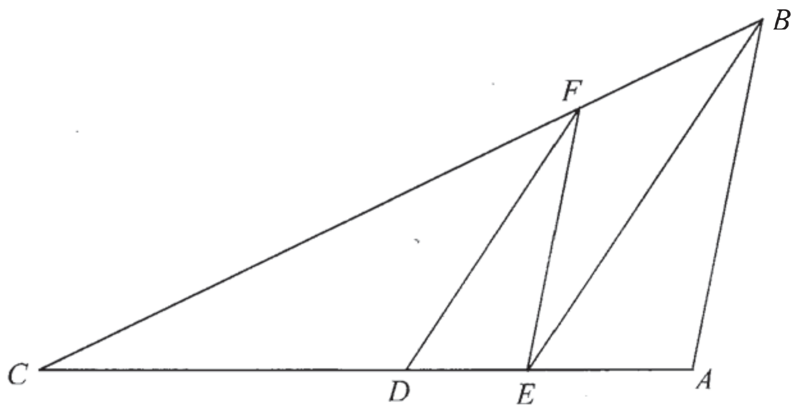
Hence the answer is $182+2\times 210=602$.

19. Answer: 49727

Since $x^2-15x+1=0$, $x+\dfrac{1}{x}=15$. Therefore

$$\begin{aligned} x^4+\frac{1}{x^4} &= \left(x+\frac{1}{x}\right)^4-4\left(x^2+\frac{1}{x^2}\right)-6 \\ &= \left(x+\frac{1}{x}\right)^4-4\left(x+\frac{1}{x}\right)^2+8-6 \\ &= 15^4-4\times 15^2+2=49727. \end{aligned}$$

20. Answer: 24



Since BE bisects $\angle ABC$, we have $AE : EC = AB : BC = 1 : 4$. Furthermore, since $EF /\!/ AB$ and $DF /\!/ EB$, we see that DF bisects $\angle EFC$. Hence $DE : DC = 1 : 4$. Let $AE = x$ and $DE = y$. Then we have $x + y = 13.5$ and $4x = 5y$. Solving the equations yields $x = 7.5$ and $y = 6$. It follows that $CD = 4y = 24$.

21. Answer: 89440

The number of such ordered triples (x, y, z) with $x = y$ is

$$\binom{65}{2}=2080.$$

The number of such ordered triples (x, y, z) with $x \neq y$ is

$$2\times\binom{65}{3}=87360.$$

Hence the answer is $2080 + 87360 = 89440$.

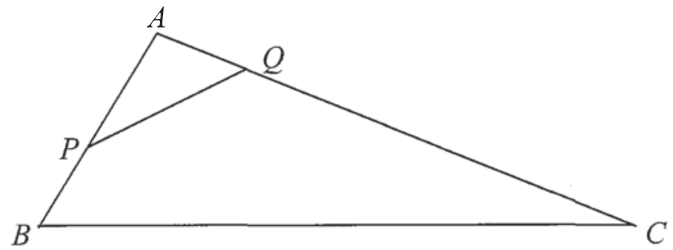
22. Answer: 19901

First note that $\dfrac{1}{a_{n+1}a_n} - \dfrac{1}{a_n a_{n-1}} = \dfrac{1+na_{n-1}a_n}{a_{n-1}a_n} - \dfrac{1}{a_n a_{n-1}} = n$. Therefore

$$\sum_{n=1}^{199}\left(\frac{1}{a_{n+1}a_n} - \frac{1}{a_n a_{n-1}}\right) = \sum_{n=1}^{199} n = \frac{199\times 200}{2} = 19900.$$

Hence $\dfrac{1}{a_{200}a_{199}} = 1 + 19900 = 19901$.

23. Answer: 6



We have $\cos A = \dfrac{AB^2 + AC^2 - BC^2}{2(AB)(AC)} = \dfrac{5^2 + 10^2 - 13^2}{2(5)(10)} = -\dfrac{11}{25}$.

Let $AP = x$ cm and $AQ = y$ cm. Since area of $\Delta APQ = \dfrac{1}{2}xy\sin A$ and area of $\Delta ABC = \dfrac{1}{2}(AB)(AC)\sin A = \dfrac{1}{2}(5)(10)\sin A$, we obtain $\dfrac{xy}{50} = \dfrac{1}{4}$, that is, $xy = \dfrac{25}{2}$.

Hence

$$PQ^2 = x^2 + y^2 - 2xy\cos A = x^2 + \left(\frac{25}{2x}\right)^2 - 25\left(-\frac{11}{25}\right)$$

$$= x^2 + \frac{625}{4x^2} + 11 \geq 2\sqrt{x^2 \cdot \frac{625}{4x^2}} + 11 = 25 + 11 = 36.$$

Consequently, $PQ \geq 6$, with the equality attained when $x = y = \dfrac{5}{\sqrt{2}}$.

24. Answer: 5

Since $x + y = 9 - z$, $xy = 24 - z(x+y) = 24 - z(9-z) = z^2 - 9z + 24$. Now note that x and y are roots of the quadratic equation $t^2 + (z-9)t + (z^2 - 9z + 24) = 0$. As x and y are real, we have $(z-9)^2 - 4(z^2 - 9z + 24) \geq 0$, which simplified to $z^2 - 6z + 5 \leq 0$. Solving the inequality yields $1 \leq z \leq 5$. When $x = y = 2$, $z = 5$. Hence the largest possible value of z is 5.

25. Answer: 357

First put the six 1's in one sequence. Then there are 7 gaps before the first 1, between two adjacent 1's and after the last 1. For each such gap, we can put a single 0 or double 0's (that is, 00).

If there are exactly i double 0's, then there are exactly $6-2i$ single 0's, where $i = 0, 1, 2, 3$. Therefore the number of such binary sequences with exactly i double 0's is $\binom{7}{i}\binom{7-i}{6-2i}$. Hence the answers is $\sum_{i=0}^{3}\binom{7}{i}\binom{7-i}{6-2i} = 357$.

26. Answer: 95

$$\frac{\cos 100^\circ}{1-4\sin 25^\circ \cos 25^\circ \cos 50^\circ} = \frac{\cos 100^\circ}{1-2\sin 50^\circ \cos 50^\circ} = \frac{\cos^2 50^\circ - \sin^2 50^\circ}{(\cos 50^\circ - \sin 50^\circ)^2}$$
$$= \frac{\cos 50^\circ + \sin 50^\circ}{\cos 50^\circ - \sin 50^\circ} = \frac{1+\tan 50^\circ}{1-\tan 50^\circ}$$
$$= \frac{\tan 45^\circ + \tan 50^\circ}{1-\tan 45^\circ \tan 50^\circ} = \tan 95^\circ.$$

Hence $x = 95$.

27. Answer: 223

$$\log_{\frac{x}{9}}\left(\frac{x^2}{3}\right) < 6 + \log_3\left(\frac{9}{x}\right)$$
$$\Leftrightarrow \frac{\log_3\left(\frac{x^2}{3}\right)}{\log_3\left(\frac{x}{9}\right)} < 6 + \log_3 9 - \log_3 x$$
$$\Leftrightarrow \frac{\log_3 x^2 - \log_3 3}{\log_3 x - \log_3 9} < 6 + \log_3 9 - \log_3 x.$$

Let $u = \log_3 x$. Then the inequality becomes $\frac{2u-1}{u-2} < 8 - u$, which is equivalent to $\frac{u^2-8u+15}{u-2} < 0$. Solving the inequality gives $u < 2$ or $3 < u < 5$, that is, $\log_3 x < 2$ or $3 < \log_3 x < 5$. It follows that $0 < x < 9$ or $27 < x < 243$. Hence there are 223 such integers.

28. Answer: 79

First observe that

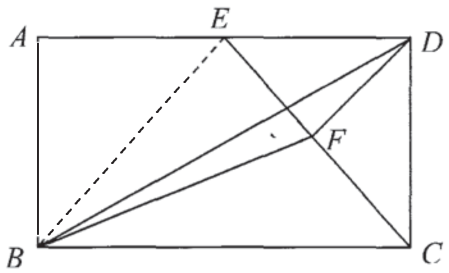
$$\frac{1}{x\sqrt{x+2}+(x+2)\sqrt{x}} = \frac{1}{\sqrt{x}\cdot\sqrt{x+2}}\left(\frac{1}{\sqrt{x}+\sqrt{x+2}}\right)$$
$$= \frac{1}{\sqrt{x}\cdot\sqrt{x+2}}\cdot\frac{\sqrt{x+2}-\sqrt{x}}{(x+2)-x}$$
$$= \frac{1}{2}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+2}}\right).$$

Therefore

$$\frac{1}{9\sqrt{11}+11\sqrt{9}}+\frac{1}{11\sqrt{13}+13\sqrt{11}}+\cdots+\frac{1}{n\sqrt{n+2}+(n+2)\sqrt{n}}$$
$$=\frac{1}{2}\left(\frac{1}{\sqrt{9}}-\frac{1}{\sqrt{11}}\right)+\frac{1}{2}\left(\frac{1}{\sqrt{11}}-\frac{1}{\sqrt{13}}\right)+\cdots+\frac{1}{2}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+2}}\right)=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{\sqrt{n+2}}\right).$$

Now $\frac{1}{2}\left(\frac{1}{3}-\frac{1}{\sqrt{n+2}}\right)=\frac{1}{9}$ yields $n = 79$.

29. Answer: 96



Let S be the area of rectangle $ABCD$. Then we have

$$\text{area of } \Delta CDF = \frac{1}{2}\times \text{area of } \Delta CDE = \frac{1}{2}\times\frac{1}{4}S = \frac{1}{8}S.$$

Next we have area of $\Delta BCF = \frac{1}{2}\times$ area of $\Delta BCE = \frac{1}{2}\times\frac{1}{2}S = \frac{1}{4}S.$

Now

$12 =$ area of $\Delta BDF =$ area of ΔBCD – area of ΔBCF – area of ΔCDF

$= \frac{1}{2}S-\frac{1}{4}S-\frac{1}{8}S=\frac{1}{8}S.$

Hence the area of rectangle $ABCD = 96$ cm^2.

30. Answer: 11754

First note that if every digit in the 6-digit number appears at least twice, then there cannot be four distinct digits in the number. In other words, the number can only be formed by using one digit, two distinct digits or three distinct digits respectively. Therefore we consider three cases.

Case 1: The 6-digit number is formed by only one digit.

Then the number of such 6-digit numbers is clearly 9.

Case 2: The 6-digit number is formed by two distinct digits.

First, the number of such 6-digit numbers formed by two given digits i and j, where $1 \le i < j \le 9$, is

$$\binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 50.$$

Next, the number of such 6-digit numbers formed by 0 and a given digit i, where $1 \le i \le 9$, is

$$\binom{5}{2} + \binom{5}{3} + \binom{5}{4} = 25.$$

Therefore the total number of such 6-digit numbers formed by two distinct digits is

$$\binom{9}{2} \times 50 + 9 \times 25 = 2025.$$

Case 3: The 6-digit number is formed by three distinct digits.

First, the number of such 6-digit numbers formed by three given digits i, j and k, where $1 \le i < j < k \le 9$, is

$$\binom{6}{2} \cdot \binom{4}{2} = 90.$$

Next, the number of such 6-digit numbers formed by 0 and two given digits i and j, where $1 \le i < j \le 9$, is

$$\binom{5}{2} \cdot \binom{4}{2} = 60.$$

Therefore the total number of such 6-digit numbers formed by three distinct digits is

$$\binom{9}{3} \times 90 + \binom{9}{2} \times 60 = 9720.$$

Hence the answer is 9 + 2025 + 9720 = 11754.

31. Answer: 234

Since $27x + 35y \le 945$, we have $y \le \dfrac{945-27x}{35}$. It follows that

$$xy \le \frac{945x-27x^2}{35} = \frac{27}{35}(35x-x^2) = \frac{27}{35}\left(\left(\frac{35}{2}\right)^2 - \left(x-\frac{35}{2}\right)^2\right).$$

Therefore, if $\left|x-\dfrac{35}{2}\right| \ge \dfrac{5}{2}$, that is, if $x \ge 20$ or $x \le 15$, then

$$xy \le \frac{27}{35}\left(\left(\frac{35}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right) < 231.4.$$

If $x = 16$, then $y \le \dfrac{945-27(16)}{35} \le 14.7$. Thus $y \le 14$, and $xy \le 224$.
Similarly, if $x = 17$, then $y \le 13$, and $xy \le 221$.
If $x = 18$, then $y \le 13$, and $xy \le 234$.
If $x = 19$, then $y \le 12$, and $xy \le 228$.
In conclusion, the maximum value of xy is 234, which is attained at $x = 18$ and $y = 13$.

32. Answer: 65520

Note that

$$(1+x^5+x^7+x^9)^{16} = \sum_{i=0}^{16}\binom{16}{i}x^{5i}(1+x^2+x^4)^i.$$

It is clear that if $i > 5$ or $i < 4$, then the coefficient of x^{29} in the expansion of $x^{5i}(1+x^2+x^4)^i$ is 0. Note also that if i is even, then the coefficient of x^{29} in the expansion of $x^{5i}(1+x^2+x^4)^i$ is also 0. Thus we only need to determine the coefficient of x^{29} in the expansion of $\dbinom{16}{i}x^{5i}(1+x^2+x^4)^i$ for $i = 5$.

When $i = 5$, we have

$$\begin{aligned}\binom{16}{i}x^{5i}(1+x^2+x^4)^i &= \binom{16}{5}x^{25}(1+x^2+x^4)^5\\ &= \binom{16}{5}x^{25}\sum_{j=0}^{5}\binom{5}{j}(x^2+x^4)^j\\ &= \binom{16}{5}x^{25}\sum_{j=0}^{5}\binom{5}{j}x^{2j}(1+x^2)^j.\end{aligned}$$

It is clear that the coefficient of x^4 in the expansion of $\sum_{j=0}^{5}\binom{5}{j}x^{2j}(1+x^2)^j$ is

$\binom{5}{1}+\binom{5}{2}=15$. Hence the answer is $\binom{16}{5}\times 15 = 65520$.

33. Answer: 401

For each n = 1, 2, 3, ..., since $d_n = \gcd(a_n, a_{n+1})$, we have $d_n \mid a_n$ and $d_n \mid a_{n+1}$. Thus $d_n \mid a_{n+1} - a_n$, that is, $d_n \mid (n+1)^2 + 100 - (n^2+100)$, which gives $d_n \mid 2n+1$. Hence $d_n \mid 2(n^2+100) - n(2n+1)$, and we obtain $d_n \mid 200 - n$. It follows that $d_n \mid 2(200-n) + 2n + 1$, that is, $d_n \mid 401$. Consequently, $1 \le d_n \le 401$ for all positive integers n.

Now when $n = 200$, we have $a_n = a_{200} = 200^2 + 100 = 401 \times 100$ and $a_{n+1} = a_{201} = 201^2 + 100 = 401 \times 101$. Therefore $d_{200} = \gcd(a_{200}, a_{201}) = 401$. Hence it follows that the maximum value of d_n when n ranges over all positive integers is 401, which is attained at $n = 200$.

34. Answer: 441

First we determine a_{2008} and a_{2009}. Suppose that $a_{2008} = \overline{x_1x_2x_3x_4x_5x_6x_7x_8}$, where the x_i's are distinct digits in $\{1,2,3,4,5,6,7,8\}$.

Let $A = \{a_k : k = 1, 2, \ldots, 40320\}$.

Since 7! = 5040 > 2008, we deduce that $x_1 = 1$, as there are more than 2008 numbers in A such that the first digit is 1.

As 2 × 6! < 2008 < 3 × 6!, we have $x_2 = 4$, as there are less than 2008 numbers in A such that the first digit is 1 and the second digit is 2 or 3, but there are more than 2008 numbers in A such that the first digit is 1 and the second digit is 2, 3 or 4. Similarly, since 2 × 6! + 4 × 5! < 2008 < 2 × 6! + 5 × 5!, we see that the third digit x_3 is 7. By repeating the argument and using the inequalities

2 × 6! + 4 × 5! + 3 × 4! < 2008 < 2 × 6! + 4 × 5! + 4 × 4! and

2004 = 2 × 6! + 4 × 5! + 3 × 4! + 2 × 3! < 2008 < 2 × 6! + 4 × 5! + 3 × 4! + 3 × 3!,

we obtain $x_4 = 6$, $x_5 = 5$. Note also that among the numbers in A of the form 1476****, the digit 5 first appears as the fifth digit in a_{2005} if the numbers are arranged in increasing order. Consequently, as the last three digits are 2, 3 and 8, we must have $a_{2005} = 14765238$. It follows that $a_{2006} = 14765283$, $a_{2007} = 14765328$, $a_{2008} = 14765382$, and $a_{2009} = 14765823$. Hence $a_{2009} - a_{2008} = 14765823 - 14765382 = 441$.

35. Answer: 24

Write $u = \log_{10} x$. Then $\log_{10}\dfrac{100}{x} = 2 - u$. Since $a = \lfloor \log_{10} x \rfloor$, we have

$$u = a + \gamma \quad \text{for some } 0 \le \gamma < 1. \qquad (1)$$

Similarly, since $b = \lfloor 2 - u \rfloor$, we have

$$2 - u = b + \delta \quad \text{for some } 0 \le \delta < 1. \qquad (2)$$

Then $0 \le \gamma + \delta < 2$. Since $\gamma + \delta = u - a + (2 - u - b) = 2 - a - b$ is an integer, it follows that $\gamma + \delta = 0$ or $\gamma + \delta = 1$.

Case 1: $\gamma + \delta = 0$.

Then $\gamma = 0$ and $\delta = 0$, since $\gamma \ge 0$ and $\delta \ge 0$. Therefore

$$\begin{aligned} 2a^2 - 3b^2 &= 2u^2 - 3(2-u)^2 \\ &= -u^2 + 12u - 12 \\ &= 24 - (u-6)^2 \le 24, \end{aligned}$$

and the maximum value is attained when $u = 6$.

Case 2: $\gamma + \delta = 1$.

Then we must have $0 < \gamma, \delta < 1$ by (1) and (2). Also, by (1) and (2), we have $b = \lfloor 2 - u \rfloor = \lfloor 2 - a - \gamma \rfloor = 1 - a$. Thus

$$\begin{aligned} 2a^2 - 3b^2 &= 2a^2 - 3(1-a)^2 \\ &= -a^2 + 6a - 3 \\ &= 6 - (a-3)^2 \le 6. \end{aligned}$$

Hence the largest possible value of $2a^2 - 3b^2$ is 24, when $x = 10^6$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section, Round 2)

Saturday, 27 June 2009 **0930-1230**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let M and N be points on sides AB and AC of triangle ABC respectively. If

$$\frac{BM}{MA} + \frac{CN}{NA} = 1,$$

show that MN passes through the centroid of ABC.

2. Find all pairs of positive integers n, m that satisfy the equation $3 \cdot 2^m + 1 = n^2$.

3. L et A be an n-element subset of $\{1, 2, \ldots, 2009\}$ with the property that the difference between any two numbers in A is not a prime number. Find the largest possible value of n. Find a set with this number of elements. (Note: 1 is not a prime number.)

4. Let $a, b, c > 0$ such that $a + b + c = 1$. Show that if if $x_1, x_2, \ldots, x_5$ are positive real numbers such that $x_1 x_2 \ldots x_5 = 1$, then

$$(ax_1^2 + bx_1 + c)(ax_2^2 + bx_2 + c) \cdots (ax_5^2 + bx_5 + c) \geq 1.$$

5. In an archery competition, there are 30 contestants. The target is divided in two zones. A hit at zone 1 is awarded 10 points while a hit at zone 2 is awarded 5 points. No point is awarded for a miss. Each contestant shoots 16 arrows. At the end of the competition statistics show that more than 50% of the arrows hit zone 2. The number of arrows that hit zone 1 and miss the target are equal. Prove that there are two contestants with the same score.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section, Round 2 solutions)

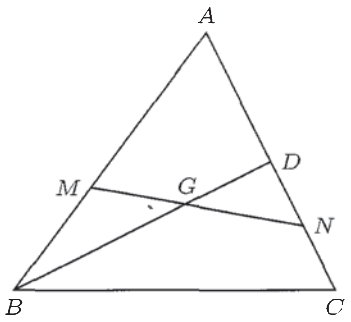
1. Let D be the midpoint of AC. Since $\frac{CN}{NA} < 1$, N lies in the segment CD. Let G be the intersection of BD and MN. By Menelaus' Theorem applied to the line MN and triangle ABD,

$$\frac{DG}{GB} \cdot \frac{BM}{MA} \cdot \frac{AN}{ND} = 1.$$

Thus

$$\begin{aligned}\frac{BG}{GD} &= \frac{BM}{MA} \cdot \frac{AN}{ND} = \left(1 - \frac{CN}{NA}\right) \cdot \frac{AN}{ND} \\ &= \frac{NA - CN}{ND} = \frac{(2CD - CN) - CN}{ND} \\ &= \frac{2ND}{ND} = 2.\end{aligned}$$

Therefore, G is the centroid of ABC.



2. We have $n^2 \equiv 1 \pmod 3$. Thus $n = 3k + 1$ or $3k + 2$ for some nonnegative integer k.

(i) $n = 3k + 1$. After simplifying, we have $2^m = 3k^2 + 2k = k(3k + 2)$. Thus k and $3k + 2$ are both powers of 2. It is clear that $k = 2$ is a solution and $k = 1$ is not. If $k = 2^p$, where $p \geq 2$, then $3k + 2 = 2(3 \cdot 2^{p-1} + 1)$ is not a power of 2 as $3 \cdot 2^{p-1} + 1$ is odd. We have one solution: $n = 7, m = 4$.

(ii) $n = 3k + 2$: Again we have $2^m = 3k^2 + 4k + 1 = (3k + 1)(k + 1)$ and both $k + 1$ and $3k + 1$ must be powers of 2. Both $k = 0, 1$ are solutions. When $k = 0$, $m = 0$, which is not admissible. For $k > 1$, we have $3k + 1 = 2k + (k + 1) > 2k + 2$ and therefore $4(k + 1) > 3k + 1 > 2(k + 1)$. Hence if $k + 1 = 2^p$ for some positive integer p, then

$2^{p+2} > 3k+1 > 2^{p+1}$ and we conclude that $3k+1$ cannot be a power of 2. Thus there is one solution in this case: $(n,m) = (5,3)$.

et A be an n-element subset of $\{1,2,\ldots,2009\}$ with the property that the difference between any two numbers in A is not a prime number. Find the largest possible value of n. Find a set with this number of elements. (Note: 1 is not a prime number.)

3. If $n \in A$, then $n+i \notin A$, $i = 2,3,5,7$. Among $n+1, n+4, n+6$ at most one can be in A. Thus among any 8 consecutive integers, at most 2 can be in S. Hence $|A| \leq 2\lceil 2009/8 \rceil = 504$. Such a set is $\{4k+1 : k = 0,1,\ldots,502\}$.

4. We give a proof of the general case. Consider the expansion of

$$(ax_1^2 + bx_1 + c)(ax_2^2 + bx_2 + c)\cdots(ax_n^2 + bx_n + c).$$

The term in $a^i b^j c^k$, where $i+j+k = n$ is

$$a^i b^j c^k[(x_1x_2\ldots x_i)^2(x_{i+1}x_{i+2}\ldots x_{i+j}) + \cdots].$$

There are altogether $\binom{n}{i}\binom{n-i}{j}$ terms in the summation. (We choose i factors from which we take ax_t^2. From the remaining $n-i$ factors, we choose j to take the terms bx_s.) By symmetry, the number of terms containing x_i^2 is a constant, as is the number of terms containing the term x_i. Thus, when the terms in the summation are multiplied together, we get $(x_1x_2\ldots x_n)^p = 1$ for some p. (For our purpose, it is not necessary to compute p. In fact $p = 2\binom{n-1}{i-1}\binom{n-i}{j} + \binom{n-1}{j-1}\binom{n-j}{i} = \frac{2i+j}{n}\binom{n}{i}\binom{n-i}{j}$.) By the AM-GM inequality, we have

$$a^i b^j c^k[(x_1x_2\ldots x_i)^2(x_{i+1}x_{i+2}\ldots x_{i+j}) + \cdots] \geq a^i b^j c^k \binom{n}{i}\binom{n-i}{j}.$$

Hence

$$(ax_1^2 + bx_1 + c)\cdots(ax_n^2 + bx_n + c) \geq \sum_{i+j+k=n} a^i b^j c^k \binom{n}{i}\binom{n-i}{j} = (a+b+c)^n = 1.$$

5. The number of arrows that that hit zone 1 is $< 30 \cdot 16/4 = 120$. If contestant i hits zone 1 a_i times, zone 2 b_i times and miss the target c_i times, then the total score is $10a_i + 5b_i = 5a_i + 5(a_i + b_i) = 5a_i + 5(16 - c_i) = 80 + 5(a_i - c_i)$. Suppose the scores are all distinct, then the 30 numbers $a_i - c_i$ must all be distinct. By the pigeonhole principle, half of these 30 numbers are either positive or negative. We consider the "positive" case. Without loss of generality, let $a_i - c_i > 0$ for $i = 1,\ldots,15$. Then $a_i - c_i \geq i$. Therefore $a_i \geq i$. Hence $a_1 + \cdots + a_{15} \geq 120$. But $a_1 + \cdots + a_{30} < 120$, and we have a contradiction. The "negative" case is similar.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 1)

Wednesday, 3 June 2009 **0930 – 1200**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

1. The expression $1000 \sin 10^0 \cos 20^0 \cos 30^0 \cos 40^0$ can be simplified as $a \sin b^0$, where a and b are positive integers with $0 < b < 90$. Find the value of $100a + b$.

2. Let A_1, A_2, A_3, A_4, A_5 and A_6 be six points on a circle in this order such that $\widehat{A_1A_2} = \widehat{A_2A_3}$, $\widehat{A_3A_4} = \widehat{A_4A_5}$ and $\widehat{A_5A_6} = \widehat{A_6A_1}$, where $\widehat{A_1A_2}$ denotes the arc length of the arc A_1A_2 etc. It is also known that $\angle A_1A_3A_5 = 72^{\circ}$. Find the size of $\angle A_4A_6A_2$ in degrees.

3. Let $P_1, P_2, \ldots, P_{41}$ be 41 distinct points on the segment BC of a triangle ABC, where AB = AC = 7. Evaluate the sum $\sum_{i=1}^{41}(\mathrm{AP}_i^2 + \mathrm{P}_i\mathrm{B} \cdot \mathrm{P}_i\mathrm{C})$.

4. Determine the largest value of x for which

$$\left|x^2 - 11x + 24\right| + \left|2x^2 + 6x - 56\right| = \left|x^2 + 17x - 80\right|.$$

5. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ be a polynomial in x where a_0, a_1, a_2, a_3, a_4 are constants and $a_5 = 7$. When divided by $x - 2004$, $x - 2005$, $x - 2006$, $x - 2007$ and $x - 2008$, f(x) leaves a remainder of 72, -30, 32, -24 and 24 respectively. Find the value of f(2009).

6. Find the value of $\dfrac{\sin 80^o}{\sin 20^o} - \dfrac{\sqrt{3}}{2\sin 80^o}$.

7. Determine the number of 8-digit positive integers such that after deleting any one digit, the remaining 7-digit number is divisible by 7.

8. It is given that $\sqrt{a}-\sqrt{b}=20$, where a and b are real numbers. Find the maximum possible value of $a - 5b$.

9. Let ABC be a triangle with sides AB = 7, BC = 8 and AC = 9. A unique circle can be drawn touching the side AC and the lines BA produced and BC produced. Let D be the centre of this circle. Find the value of BD^2.

10. If $x=\frac{1}{2}\left(\sqrt[3]{2009}-\frac{1}{\sqrt[3]{2009}}\right)$, find the value of $\left(x+\sqrt{1+x^2}\right)^3$.

11. Let $S = \{1, 2, 3,\ldots\ldots, 30\}$. Determine the number of vectors (x, y, z, w) with $x, y, z, w \in S$ such that $x < w$ and $y < z < w$.

12. Let f(n) be the number of 0's in the decimal representation of the positive integer n. For example, f(10001123) = 3 and f(1234567) = 0. Find the value of

f(1) + f(2) + f(3) + ……+ f(99999).

13. It is given that k is a positive integer not exceeding 99. There are no natural numbers x and y such that $x^2 - ky^2 = 8$. Find the difference between the maximum and minimum possible values of k.

14. Let $S = \{1, 2, 3, 4,\ldots, 16\}$. In each of the following subsets of S,

$\{6\}$, $\{1, 2, 3\}$, $\{5, 7, 9, 10, 11, 12\}$, $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

the sum of all the elements is a multiple of 3. Find the total number of non-empty subsets A of S such that the sum of all elements in A is a multiple of 3.

15. A function $f : \mathbf{R} \rightarrow \mathbf{R}$ satisfies the relation $f(x)f(y) = f(2xy + 3) + 3f(x + y) - 3f(x) + 6x$, where

$x, y \in \mathbf{R}$. Find the value of f(2009).

16. Let $\{a_n\}$ be a sequence of positive integers such that $a_1 = 1$, $a_2 = 2009$ and for $n \geq 1$,

$a_{n+2}a_n - a_{n+1}^2 - a_{n+1}a_n = 0$. Determine the value of $\frac{a_{993}}{100a_{991}}$.

17. Determine the number of ways of tiling a 4x9 rectangle by tiles of size 1x2.

18. Find the number of 7-digit positive integers such that the digits from left to right are non-increasing. (Examples of 7-digit non-increasing numbers are 9998766 and 5555555; An example of a number that is NOT non-increasing is 7776556)

19. Determine the largest prime number less than 5000 of the form $a^n - 1$, where a and n are positive integers, and n is greater than 1.

20. Determine the least constant M such that

$$\frac{x_1}{x_1+x_2}+\frac{x_2}{x_2+x_3}+\frac{x_3}{x_3+x_4}+\cdots+\frac{x_{2009}}{x_{2009}+x_1}<M,$$

for any positive real numbers $x_1, x_2, x_3, \ldots, x_{2009}$.

21. Six numbers are randomly selected from the integers 1 to 45 inclusive. Let p be the probability that at least three of the numbers are consecutive. Find the value of $\lfloor 1000p \rfloor$. (Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x).

22. Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{2}{\pi} \tan^{-1}\left(\frac{2}{(2k+1)^2}\right)$.

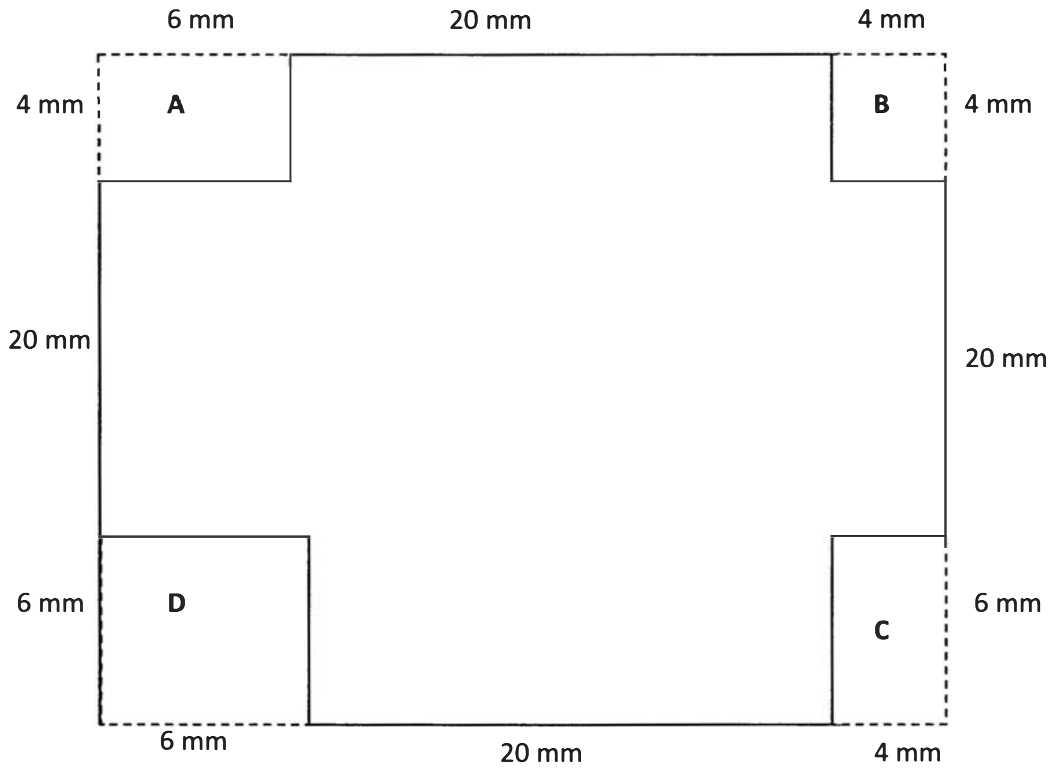
23. Determine the largest prime factor of the sum $\displaystyle\sum_{k=1}^{11} k^5$.

24. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $x_1 = 3, x_2 = 24$ and

$$x_{n+2} = \frac{1}{4}x_{n+1} + \frac{3}{4}x_n$$

for every positive integers n. Determine the value of $\lim\limits_{n\to\infty} x_n$.

25. A square piece of graph paper of side length 30 mm contains 900 smallest squares each of side length 1mm each. Its four rectangular corners, denoted by A, B, C, D in clockwise order, are cut away from the square piece of graph paper. The resultant graph paper, which has the shape of a cross, is shown in the figure below. Let N denote the total number of rectangles, **excluding** all the squares which are contained in the resultant graph paper. Find the value of $\frac{1}{10}N$.



Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 1 Solution)

1. Answer: 12560

$$\sin 10^0 \cos 20^0 \cos 30^0 \cos 40^0 = \frac{2\sin 10^0 \cos 10^0 \cos 20^0 \cos 30^0 \cos 40^0}{2\cos 10^0}$$

$$= \frac{\sin 20^0 \cos 20^0 \cos 30^0 \cos 40^0}{2\cos 10^0}$$

$$= \frac{\sin 40^0 \cos 40^0 \cos 30^0}{4\cos 10^0}$$

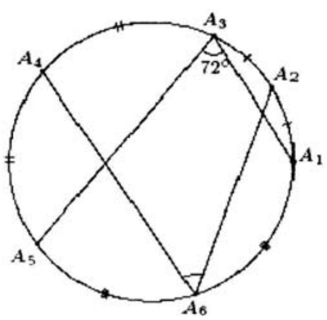
$$= \frac{\sin 80^0 \cos 30^0}{8\cos 10^0}$$

$$= \frac{\cos 10^0 \sin 60^0}{8\cos 10^0}$$

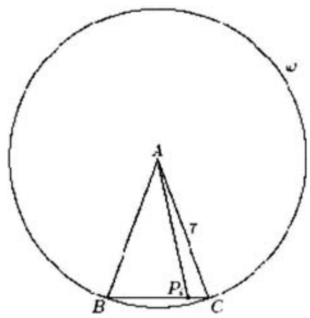
Hence $1000\sin 10^0 \cos 20^0 \cos 30^0 \cos 40^0 = 125\sin 60^0$, showing that $a = 125$ and $b = 60$. So, $100a + b = 12560$.

2. Answer: 54.

First, observe that $\angle A_1A_6A_5 = 180^o - 72^o = 108^o$. Hence $\angle A_4A_6A_2 = \angle A_1A_6A_5 - \angle A_1A_6A_2 - \angle A_4A_6A_5 = 108^o - \angle A_2A_6A_3 - \angle A_3A_6A_4 = 108^o - \angle A_4A_6A_2$. Thus, $2\angle A_4A_6A_2 = 108^o$, resulting in $\angle A_4A_6A_2 = 54^o$.



3. Answer: 2009



Construct a circle ω centred at A with radius AB = AC = 7. The power of P_i with respect to ω is $P_iA^2 - 7^2$, which is also equal to $-BP_i \cdot P_iC$. Thus, $P_iA^2 + BP_i \cdot P_iC = 7^2 = 49$. Therefore, the sum equals 49 x 41 = 2009.

4. Answer: 8

Since $2x^2 + 6x - 56 - (x^2 - 11x + 24) = x^2 + 17x - 80$, the given equation holds if and only if

$$\left(x^2 - 11x + 24\right)\left(2x^2 + 6x - 56\right) \le 0,$$

Since $|a - b| = |a| + |b|$ if and only if $ab \le 0$. The above inequality reduces to

$$(x-3)(x-8)(x-4)(x+7) \le 0.$$

Since

$$\{x : (x-3)(x-8)(x-4)(x+7) \le 0\} = [-7,3] \cup [4,8],$$

we conclude that the largest value of x is 8.

5. Answer: 1742.

We have

$$f(x) = \frac{(x-2005)(x-2006)(x-2007)(x-2008)}{(-1)(-2)(-3)(-4)}(72) + \frac{(x-2004)(x-2006)(x-2007)(x-2008)}{(1)(-1)(-2)(-3)}(-30) + \frac{(x-2004)(x-2005)(x-2007)(x-2008)}{(2)(1)(-1)(-2)}(32) + \frac{(x-2004)(x-2005)(x-2006)(x-2008)}{(3)(2)(1)(-1)}(-24) + \frac{(x-2004)(x-2005)(x-2006)(x-2007)}{(4)(3)(2)(1)}(24) + 7(x-2004)(x-2005)(x-2006)(x-2007)(x-2008),$$

So that f(2009) = 1742.

6. Answer: 2

$$\frac{\sin 80^o}{\sin 20^o}-\frac{\sqrt{3}}{2\sin 80^o}=\frac{\sin 80^o}{\sin 20^o}-\frac{\sin 60^o}{\sin 80^o}$$

$$=\frac{\sin^2 80^o-\sin 20^o\sin 60^o}{\sin 20^o\sin 80^o}=\frac{1-\cos 160^o+\cos 80^o-\cos 40^o}{2\sin 20^o\sin 80^o}$$

$$=\frac{1-\cos 40^o+\cos 80^o-\cos 160^o}{2\sin 20^o\sin 80^o}$$

$$=\frac{2\sin^2 20^o+2\sin 120^o\sin 40^o}{2\sin 20^o\sin 80^o}=\frac{2\sin^2 20^o+2\sqrt{3}\sin 20^o\cos 20^o}{2\sin 20^o\sin 80^o}$$

$$=\frac{\sin 20^o+\sqrt{3}\cos 20^o}{\sin 80^o}=\frac{2\sin(20^o+60^o)}{\sin 80^o}=2$$

7. Answer: 64

Let $N=\overline{abcdefgh}$ be such a number. By deleting a and b, we get $\overline{bcdefgh}$ and $\overline{acdefgh}$ respectively. Both of them are divisible by 7, hence their difference 1000000 $(b-a)$ is also divisible by 7, therefore $b-a$ is divisible by 7. By the similar argument, $c-b$, $d-c$, $e-d$, $f-e$, $g-f$, $h-g$ are divisible by 7. In other word, all the digits of N are congruent modulo 7. If N contains digits that are greater than 7, then one can subtract 7 from each digit to get a new number N'. Then N satisfies the requirements in the question if and only if N' does. Since all the digits are congruent modulo 7, it remains to consider numbers of the form $\overline{pppppppp}$, where $p = 0, 1, 2, 3, 4, 5, 6$. By deleting a digit in this number, we get the number $\overline{ppppppp}=1111111p$, which is divisible by 7 if and only if p is the digit 0 or 7. However, the first two digits of N must be 7 since the number N has 8 digits and that any number we get by deleting a digit in N has 7 digits. On the other hand, the remaining 6 digits can be independently 0 or 7. Consequently, there are $2^6 = 64$ choices of such numbers.

8. Answer: 500

$$\sqrt{a} = \sqrt{b} + 20$$
$$a = b + 400 + 40\sqrt{b}$$
$$a - 5b = 400 + 40\sqrt{b} - 4b$$
$$a - 5b = 400 - 4(\sqrt{b} - 5)^2 + 100$$
$$a - 5b \le 500.$$

9. Answer: 224

Let the circle with centre D and radius r touch the tangent lines AC, BA produced and BC produced at the points E, F and G respectively. Then r = DE = DF = DF. Hence, triangles BDF and BDG are congruent, and hence ∠ABD = ∠CBD = ½ ∠ABC. We have

$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{8^2+7^2-9^2}{2(8)(7)} = \frac{2}{7}$, and hence $\sin\frac{B}{2} = \sqrt{\frac{1-\cos B}{2}} = \sqrt{\frac{5}{14}}$.

To find r, we have

(ABD) + (BCD) – (ACD) = (ABC),

where (ABD) denotes the area of triangle ABD, etc.

Hence $\frac{1}{2}cr + \frac{1}{2}ar - \frac{1}{2}br = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{8+9+7}{2} = 12$.

Solving, we get $r = 4\sqrt{5}$. Considering, triangle BDF, we have BD $= \frac{r}{\sin\frac{B}{2}} = 4\sqrt{14}$. Thus, we have $BD^2 = 224$.

10. Answer: 2009

Since $x = \frac{1}{2}\left(\sqrt[3]{2009} - \frac{1}{\sqrt[3]{2009}}\right)$, we have $(\sqrt[3]{2009})^2 - 2x\sqrt[3]{2009} - 1 = 0$ .We see that $\sqrt[3]{2009}$ is a root of the equation $t^2 - 2xt - 1 = 0$. Thus

$\sqrt[3]{2009} = \frac{2x+\sqrt{4x^2+4}}{2} = x + \sqrt{x^2+1}$ or $\sqrt[3]{2009} = \frac{2x-\sqrt{4x^2+4}}{2} = x - \sqrt{x^2+1} < 0$,

which is not possible. Thus $\left(x + \sqrt{1+x^2}\right)^3 = 2009$.

11. Answer: 90335

There are two cases to consider: Case (1) $x \in \{y, z\}$ and Case (2): $x \notin \{y, z\}$. For Case (1), there are $2\binom{30}{3}$ ways and for Case (2), there are $3\binom{30}{4}$ ways. Hence, total number of ways = 90335.

12. Answer: 38889

Let $S = \{1, 2, 3, \ldots, 99999\}$, and $S_i = \{n \in S\colon f(n) = i\}$ for $i \geq 0$. Thus,

$$S = \bigcup_{0 \leq i \leq 4} S_i.$$

For $0 \leq i \leq 4$, if $n \in S_i$, and n has exactly k digits in the decimal representation, then exactly we have $k - i$ digits are non-zero. Thus,

$$|S_i| = \sum_{k=i+1}^{5} \binom{k-1}{i} 9^{k-i}.$$

Then, it is clear that

$$M = |S_1| + 2|S_2| + 3|S_3| + 4|S_4| = 38889.$$

13. Answer: 96

Note that $3^2 - 1\times 1^2 = 8$ and $4^2 - 2\times 2^2 = 8$. Suppose $k \equiv 0 \pmod 3$. Note that $a^2 \equiv 0,1 \pmod 3$ for all natural numbers a. Thus, $x^2 - ky^2 \equiv 0,1 - 0 \pmod 3 \equiv 0,1 \pmod 3$ but $8 \equiv 2 \pmod 3$. Hence, there are no natural numbers x and y such that $x^2 - ky^2 = 8$ if k is a multiple of 3. Therefore, $\max\{k\} - \min\{k\} = 99 - 3 = 96$.

14. Answer: 21855.

Let $S_i = \{x \in S\colon x \equiv i \pmod 3\}$ for $i = 0,1,2$. Note that $|S_0| = 5$, $|S_1| = 6$ and $|S_2| = 5$. Let ∂ be the set of all subsets A of S such that $\sum_{x\in\partial} x$ is a multiple of 3. Note that for any $A \subseteq S$,

$$\sum_{x\in A} x = \sum_{i=0}^{2} \sum_{x\in A\cap S_i} x \equiv |A \cap S_1| + 2|A \cap S_2| \pmod 3.$$

Thus, $A \in \partial$ if and only if $|A \cap S_1| \equiv |A \cap S_2| \pmod 3$. Thus, it is clear that

$|\partial| = 2^{|S_0|}m$, where

$$m = \left\{\binom{6}{0}+\binom{6}{3}+\binom{6}{6}\right\}\left\{\binom{5}{0}+\binom{5}{3}\right\}+\left\{\binom{6}{1}+\binom{6}{4}\right\}\left\{\binom{5}{1}+\binom{5}{4}\right\}$$
$$+\left\{\binom{6}{2}+\binom{6}{5}\right\}\left\{\binom{5}{2}+\binom{5}{5}\right\} = 683.$$

Hence $|\partial| = 2^5 \times 683 = 21856$. Since we want only non-empty subsets, we have 21855.

15. Answer: 4021

Given $f(x)f(y) = f(2xy + 3) + 3f(x + y) - 3f(x) + 6x$, so if interchanging x and y we have

$f(y)f(x) = f(2xy + 3) + 3f(x + y) - 3f(y) + 6y$.
Subtracting, we have $-3f(x) + 6x = -3f(y) + 6y$ for all $x, y \in \mathbf{R}$, showing that $f(x) - 2x$ is a constant, let it be k. So, $f(x) = 2x + k$.
Substitute back to the given functional equation, we have

$(2x + k)(2y + k) = 2(2xy + 3) + k + 3[2(x + y) + k] - 3(2x + k) + 6x$

$4xy + 2kx + 2ky + k^2 = 4xy + 6 + k + 6x + 6y + 3k - 6x - 3k + 6x$
$2k(x + y) - 6(x + y) = k - k^2 + 6$
$(k - 3)(k + 2) = 2(x + y)(3 - k)$

$(k - 3)(k + 2 + 2x + 2y) = 0$ for all $x, y \in \mathbf{R}$.

Thus $k = 3$. Hence $f(2009) = 2(2009) + 3 = 4021$.

16. Answer: 89970

$$a_{n+2}a_n - a_{n+1}^2 - a_{n+1}a_n = 0 \quad \Rightarrow \quad \frac{a_{n+2}a_n - a_{n+1}^2 - a_{n+1}a_n}{a_{n+1}a_n} = 0$$

$\frac{a_{n+2}}{a_{n+1}} - \frac{a_{n+1}}{a_n} = 1$. From here, we see that $\left\{\frac{a_{n+1}}{a_n}\right\}$ is an arithmetic sequence with first term 2009 and common difference 1. Thus $\frac{a_{n+1}}{a_n} = n + 2008$, and that

$\frac{a_{993}}{a_{992}} = 992 + 2008 = 3000$ and $\frac{a_{992}}{a_{991}} = 991 + 2008 = 2999$. We therefore have

$\frac{a_{993}}{100a_{991}} = 30(2999) = 89970.$

17. Answer: 6336

Let f_n be the number of ways of tiling a 4 x n rectangle. Also, let g_n be the number of ways of tiling a 4 x n rectangle with the top or bottom two squares in the last column missing, and let h_n be the number of ways of tiling a 4 x n rectangle with the top and bottom squares in the last column missing. Set up a system of recurrence relations involving f_n , g_n and h_n by considering the ways to cover the nth column of a 4 x n rectangle. If two vertical tiles are used, then there are f_{n-1} ways. If one vertical tile and two adjacent horizontal tiles are used, then there are $2g_{n-1}$ ways. If one vertical tile and two non-adjacent horizontal tiles are used, then there are h_{n-1} ways. If four horizontal tiles are used, then there are f_{n-2} ways. Similarly, one can establish the recurrence relations for g_n and h_n. In conclusion, we obtain for $n \geq 2$,

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} + 2g_{n-1} + h_{n-1} \\ g_n &= g_{n-1} + f_{n-1} \\ h_n &= h_{n-2} + f_{n-1}, \end{aligned}$$

With initial conditions $f_0 = f_1 = g_1 = h_1 = 1$ and $h_0 = 0$. Solving f_n recursively, we obtain $f_9 =$ 6336.

18. Answer: 11439

Each 16-digit binary sequence containing exactly nine '0's and seven '1's can be matched uniquely to such a 7-digit integer or 0000000 as follows: Each '1' will be replaced by a digit from 0 to 9 in this way: the number of '0's to the right of a particular '1' indicates the value of the digit. For example, 0110000010101101 ~ 8832110 and 1111000000000111 ~ 9999000.

Thus, required number = $\binom{16}{7} - 1$.

19. Answer: 127

Since $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \cdots + a + 1)$ is a prime, then $a = 2$. Suppose $n = rs$ is a composite, then $2^{rs} - 1 = (2^r - 1)(2^{r(s-1)} + 2^{r(s-2)} + \cdots + 2^r + 1)$, where each factor on the right is greater than 1, contradicting the fact that $a^n - 1$ is a prime. Therefore n must be a prime. The largest prime n such that $2^n - 1 < 5000$ is 11. However, $2^{11} - 1 = 2047 = 23 \times 89$, which is not a prime. Since $2^7 - 1 = 127$ is a prime number, the answer to this question is 127.

20. Answer: 2008

Let $S = \frac{x_1}{x_1+x_2} + \frac{x_2}{x_2+x_3} + \frac{x_3}{x_3+x_4} + \cdots + \frac{x_{2009}}{x_{2009}+x_1}$. Then it is clear that $S > \frac{x_1}{x_1+x_{2+\cdots}+x_{2009}} + \frac{x_2}{x_1+x_2+\cdots+x_{2009}} + \cdots + \frac{x_{2009}}{x_1+x_{2+\cdots}+x_{2009}} = 1$. Next, set

$S' = \frac{x_{2009}}{x_{2009}+x_{2008}} + \frac{x_{2008}}{x_{2008}+x_{2007}} + \frac{x_{2007}}{x_{2007}+x_{2006}} + \cdots + \frac{x_1}{x_1+x_{2009}}$. By the same reasoning, $S' > 1$.

Note that $S + S' = 2009$. We claim that $S < 2008$. Suppose that $S \geq 2008$, then we must have $2009 = S + S' > 2008 + 1 = 2009$, which is absurd. Hence our claim that $S < 2008$ is true.

We shall next show that M is the least possible bound for S. Consider the numbers $x_i = a^i$, where $I = 1, 2, 3, \ldots, 2009$. Direct computation yields $S = \frac{2008}{a+1} + \frac{a^{2008}}{a^{2008}+1}$. When a is chosen to be arbitrarily close to 0, this expression gets arbitrarily close to 2008.

21. Answer: 56

The number of selections such that no two of the numbers are consecutive = The number of binary sequences containing 6 '1's and 34 '0's = $\binom{40}{6}$.

The number of selections such that exactly two of the numbers are consecutive = The number of binary sequences containing a '11', 4 '1's and 35 '0's = $\binom{5}{1}\binom{40}{5}$.

The number of selections with exactly two sets of two consecutive numbers but no three numbers are consecutive = The number of binary sequences containing 2 '11's, 2 '1's and 36 '0's = $\binom{4}{2}\binom{40}{4}$.

The number of selections with exactly three sets of two consecutive numbers but no three numbers are consecutive = The number of binary sequences containing 3 '11's and 37 '0's = $\binom{40}{3}$.

Thus, $1000p = 1000\left[\binom{45}{6} - \binom{40}{6} - \binom{5}{1}\binom{40}{5} - \binom{4}{2}\binom{40}{4} - \binom{40}{3}\right] \div \binom{45}{6} = 56.28..$

22. Answer: 1

$$\sum_{k=0}^{\infty}\frac{2}{\pi}\tan^{-1}\left(\frac{2}{(2k+1)^2}\right)$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{2}{\pi}\tan^{-1}\left(\frac{2}{(2k+1)^2}\right)$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{2}{\pi}\tan^{-1}\left(\frac{2k+2-2k}{1+(2k)(2k+2)}\right)$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{2}{\pi}\left(\tan^{-1}(2k+2)-\tan^{-1}(2k)\right)$$

$$= \lim_{n\to\infty}\frac{2}{\pi}\tan^{-1}(2n+2)=1.$$

23. Answer: 263

We consider the sum $\sum_{k=1}^{11}\left(6k^5+2k^3\right)$ and observe that

$$\sum_{k=1}^{11}\left(6k^5+2k^3\right)$$

$$=\sum_{k=1}^{11}\left\{k^3(k+1)^3-k^3(k-1)^3\right\}$$

$$=11^3\times 12^3.$$

Hence

$$\sum_{k=1}^{11}k^5$$

$$=\frac{11^3\times 12^3}{6}-\frac{2}{6}\sum_{k=1}^{11}k^3$$

$$=2\times 11^3\times 12^2-\frac{1}{3}\left(\frac{11^2\times 12^2}{4}\right)$$

$$=11^2\times 12\times\left(2\times 11\times 12-1\right)$$

$$=11^2\times 12\times 263.$$

Finally, since $\sqrt{256}=16$ and the numbers 2, 3, 5, 7, 11 and 13 do not divide 263, we conclude that 263 is the largest prime factor of $\sum_{k=1}^{11}k^5$.

24. Answer: 15

For each positive integer n, we have

$$x_{n+2} - x_{n+1} = -\frac{3}{4}(x_{n+1} - x_n)$$

and so we have $x_{n+1} - x_n = \left(-\frac{3}{4}\right)^{n-1}(x_2 - x_1) = 21\left(-\frac{3}{4}\right)^{n-1}$.

Therefore, $x_n = x_1 + 21\sum_{k=1}^{n-1}\left(-\frac{3}{4}\right)^{k-1}$ for all positive integers $n \geq 3$. --- (1)

Now, we let $n \to \infty$ in (1) to conclude that $\lim_{n\to\infty} x_n = x_1 + 21\sum_{k=1}^{\infty}\left(-\frac{3}{4}\right)^{k-1} = 3 + \frac{21}{1+0.75} = 15.$

25. Answer: 14413

The number of rectangles (including squares)

= (20 + 19 + 18 +…+2 + 1)(30 + 29 + 28 +…..+1) x 2 – $(20 + 19 + 18 +….. + 1)^2$

= 151200.

The number of squares

= (20 x 30 + 19 x 29 + 18 x 28 + ….. + 1 x 11) x 2 – $(20^2 + 19^2 + 18^2 + …. + 1^2)$

= 7070.

N = the number of rectangles less all squares = 151200 – 7070 = 144130.

Hence N / 10 = 14413.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 2)

Saturday, 4 July 2009 **0900-1330**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let O be the center of the circle inscribed in a rhombus $ABCD$. Points E, F, G, H are chosen on sides AB, BC, CD and DA respectively so that EF and GH are tangent to the inscribed circle. Show that EH and FG are parallel.

2. A palindromic number is a number which is unchanged when the order of its digits is reversed. Prove that the arithmetic progression 18, 37, ... contains infinitely many palindromic numbers.

3. For k a positive integer, define A_n for $n = 1, 2, \ldots,$ by

$$A_{n+1} = \frac{nA_n + 2(n+1)^{2k}}{n+2}, \quad A_1 = 1.$$

Prove that A_n is an integer for all $n \geq 1$, and A_n is odd if and only if $n \equiv 1$ or $2 \pmod 4$.

4. Find the largest constant C such that

$$\sum_{i=1}^{4}(x_i + \frac{1}{x_i})^3 \geq C$$

for all positive real numbers $x_1, \cdots, x_4$ such that

$$x_1^3 + x_3^3 + 3x_1x_3 = x_2 + x_4 = 1.$$

5. Find all integers x, y and z with $2 \leq x \leq y \leq z$ such that

$$xy \equiv 1 \pmod z, \quad xz \equiv 1 \pmod y, \quad yz \equiv 1 \pmod x.$$

Singapore Mathematical Society

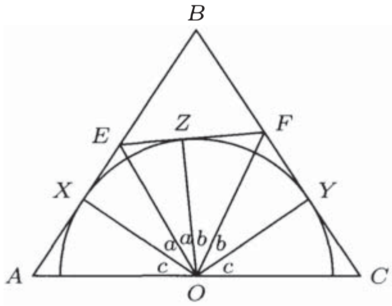
Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 2 solutions)

1. The figure shows half of the rhombus (which is an isosceles triangle), where X, Y, Z are points of tangency of the circle to the sides AB, CB and EF respectively. Note that

$$\angle XOE = \angle EOZ, \quad \angle ZOF = \angle FOY, \quad \angle AOX = \angle COY.$$

In particular, $a + b + c = 90°$.



Thus

$$\angle AEO = 90° - a = b + c = \angle COF.$$

Hence the triangles AOE and CFO are similar. It follows that $AE \cdot CF = AO^2$. Similarly, on the lower half of the rhombus, $AO^2 = AH \cdot CG$. Then $AE/AH = CG/CF$ and hence the triangles AEH and CGF are similar. Thus $\angle AEH = \angle CGF$. Since AB is parallel to CD, it follows that EH is parallel to FG.

2. Let $a_i = 18 + 19i$. We'll show that there are infinitely many i such that a_i consists of only the digit 1, i.e.

$$a_i = 18 + 19i = \frac{10^k - 1}{9}.$$

This yields $10^k \equiv 11 \pmod{19}$. Thus any positive integer of the form $\frac{10^k-1}{9}$, where $10^k \equiv 11 \pmod{19}$ is in the AP. Since $10^6 \equiv 11$ and $10^{18} \equiv 1 \pmod{19}$, we have $10^{18t+6} \equiv 11 \pmod{19}$ for any t. Thus there are infinitely many palindromic numbers.

3. We have

$$(n+2)A_{n+1} - nA_n = 2(n+1)^{2k}$$
$$(n+1)A_n - (n-1)A_{n-1} = 2(n)^{2k}$$

From these we get

$$
\begin{aligned}
(n+1)(n+2)A_{n+1} - n(n+1)A_n &= 2(n+1)^{2k+1} \\
n(n+1)A_n - (n-1)nA_{n-1} &= 2(n)^{2k+1} \\
(n+1)(n+2)A_{n+1} - (n-1)nA_{n-1} &= 2(n+1)^{2k+1} + 2(n)^{2k+1}
\end{aligned}
$$

Using this recurrence, we obtain

$$A_n = \frac{2S(n)}{n(n+1)} \quad \text{where} \quad S(n) = 1^t + 2^t + \cdots + n^t,\ t = 2k+1.$$

Since

$$2S(n) = \sum_{i=0}^{n} \left((n-i)^t + i^t\right) = \sum_{i=1}^{n} \left((n+1-i)^t + i^t\right)$$

we see that $n(n+1) \mid 2S(n)$. Thus A_n is an integer for all n.

(i) $n \equiv 1$ or $2 \pmod 4$. Then $S(n)$ is odd since it has an odd number of odd terms. Thus A_n is odd.

(ii) $n \equiv 0 \pmod 4$. Then $(n/2)^t \equiv 0 \pmod n$. Thus

$$S(n) = \sum_{i=0}^{n/2} \left((n-i)^t + i^t\right) - \left(\frac{n}{2}\right)^t \equiv 0 \pmod n.$$

Thus A_n is even.

(iii) $n \equiv 3 \pmod 4$. Then $((n+1)/2)^t \equiv 0 \pmod{n+1}$. Thus

$$S(n) = \sum_{i=1}^{(n+1)/2} \left((n+1-i)^t + i^t\right) - \left(\frac{n+1}{2}\right)^t \equiv 0 \pmod{n+1}.$$

Thus A_n is even.

4. First note that

$$
\begin{aligned}
x_1^3 + x_3^3 + 3x_1x_3 - 1 &= x_1^3 + x_3^3 - (1)^3 - 3x_1x_3(-1) \\
&= (x_1 + x_3 - 1)\left((x_1+x_3)^2 + (x_1+x_3) + 1\right) - 3x_1x_3(x_1+x_3-1) \\
&= (x_1 + x_3 - 1)[(x_1 - x_3)^2 + (x_1+1)(x_3+1)].
\end{aligned}
$$

It is equal to zero only when either $x_1 + x_3 = 1$ or $x_1 = x_3 = -1$. Thus we must have $x_1 + x_3 = 1$ as they are positive. It now suffixes to show that the following is sharp:

$$\sum_{i=1}^{2} \left(y_i + \frac{1}{y_i}\right)^3 \geq 125/4 \ \text{ when } y_1 + y_2 = 1 \ \text{ and } y_1, y_2 > 0.$$

To this end, it is clear that the function $f(x) = (x + 1/x)^3$ is convex. Thus,

$$f(x) + f(1 - x) \geq 2f(1/2) = 125/4.$$

5. The only solution is $(x, y, z) = (2, 3, 5)$.

First of all, observe that $(x, y) = (x, z) = (y, z) = 1$. Then $2 \leq x < y < z$, and combining the three given congruences we can express it as

$$xy + xz + yz - 1 \equiv 0 \pmod{x,\ y,\ \ z}.$$

Since x, y and z are pairwise coprime, we have

$$xy + xz + yz - 1 \equiv 0 \pmod{xyz}.$$

It follows that $xy + xz + yz - 1 = k(xyz)$ for some integer $k \geq 1$. Dividing by xyz, we obtain that

$$\frac{1}{z} + \frac{1}{y} + \frac{1}{x} = \frac{1}{xyz} + k > 1.$$

Since $x < y < z$, it follows that

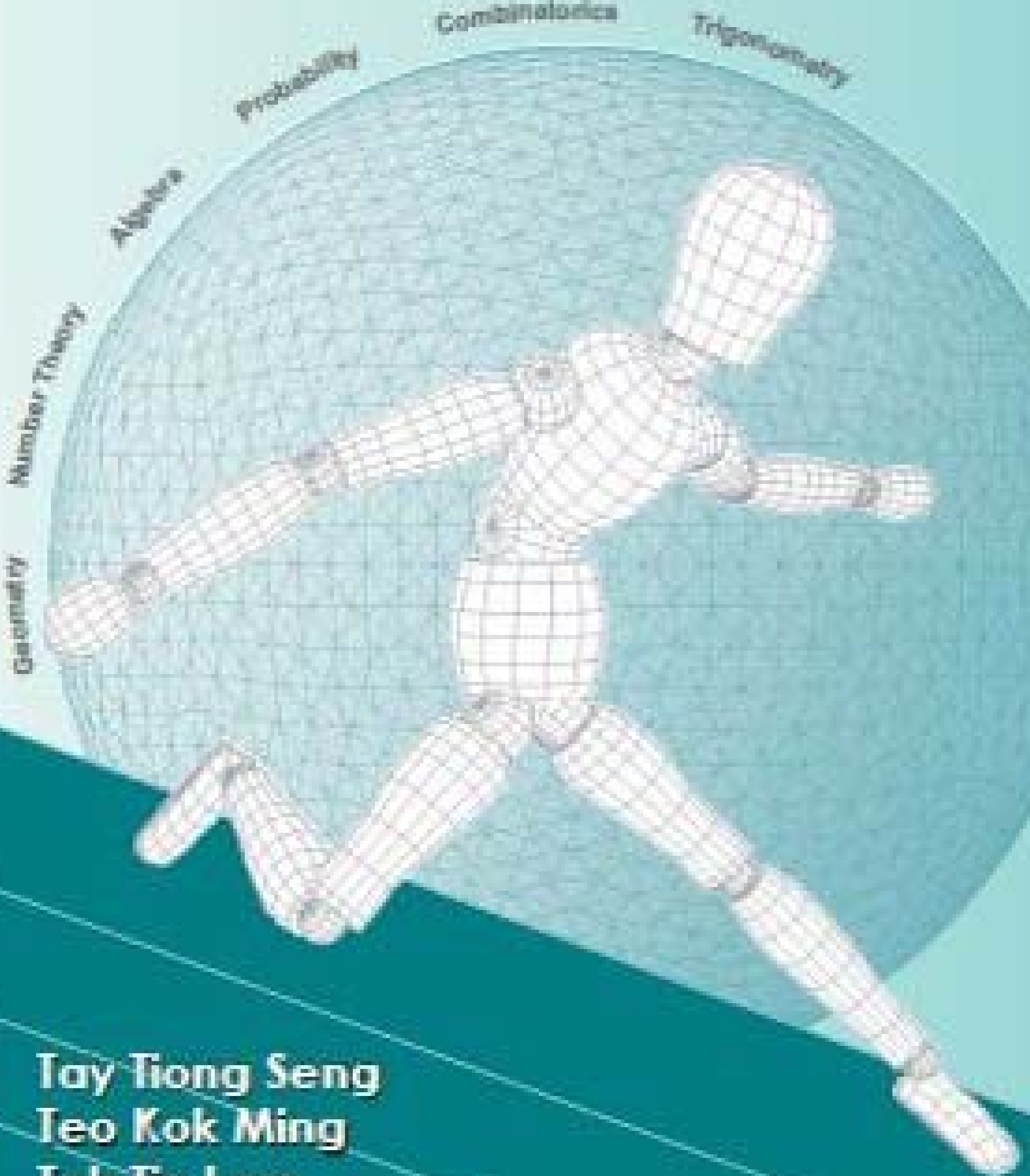
$$1 < \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{3}{x}$$

and this gives $x = 2$ as the only value. In this case, the inequalities give

$$\frac{1}{2} < \frac{1}{y} + \frac{1}{z} < \frac{2}{y},$$

which implies that $y = 3$. It follows that the only possible values of z are 4 and 5. Hence, for $2 \leq x < y < z$, the solutions are $(x, y, z) = (2, 3, 4)$ and $(2, 3, 5)$. Since 2 and 4 are not relatively prime, the only solution is $(x, y, z) = (2, 3, 5)$.

SINGAPORE MATHEMATICAL OLYMPIADS
2010



Tay Tiong Seng
Teo Kok Ming
Toh Tin Lam
Yang Yue

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Singapore Mathematical Olympiad (SMO) 2010

(Junior Section)

Tuesday, 1 June 2010 0930-1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer on the answer sheet and shade the appropriate bubbles below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Among the five real numbers below, which one is the smallest?

   (A) $\sqrt[2009]{2010}$; (B) $\sqrt[2010]{2009}$; (C) 2010; (D) $\dfrac{2010}{2009}$; (E) $\dfrac{2009}{2010}$.

2. Among the five integers below, which one is the largest?

   (A) 2009^{2010}; (B) 20092010^2; (C) 2010^{2009}; (D) $3^{(3^{(3^3)})}$; (E) $2^{10} + 4^{10} + \cdots + 2010^{10}$.

3. Among the four statements on real numbers below, how many of them are correct?

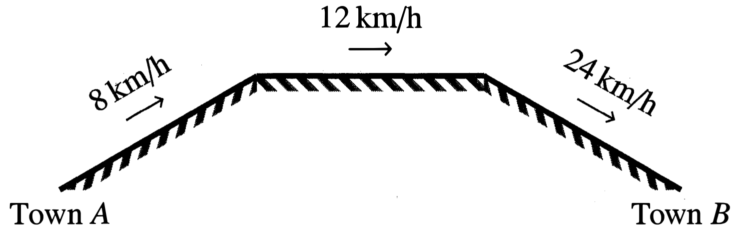
   "If $a < b$ and $a, b \neq 0$ then $\frac{1}{b} < \frac{1}{a}$"; "If $a < b$ then $ac < bc$";
   "If $a < b$ then $a + c < b + c$"; "If $a^2 < b^2$ then $a < b$".

   (A) 0; (B) 1; (C) 2; (D) 3; (E) 4.

4. What is the largest integer less than or equal to $\sqrt[3]{(2010)^3 + 3 \times (2010)^2 + 4 \times 2010 + 1}$?

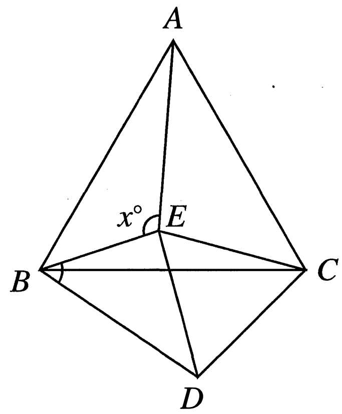
   (A) 2009; (B) 2010; (C) 2011; (D) 2012; (E) None of the above.

5. The conditions of the road between Town A and Town B can be classified as up slope, horizontal or down slope and total length of each type of road is the same. A cyclist travels from Town A to Town B with uniform speeds 8 km/h, 12 km/h and 24 km/h on the up slope, horizontal and down slope road respectively. What is the average speed of his journey?



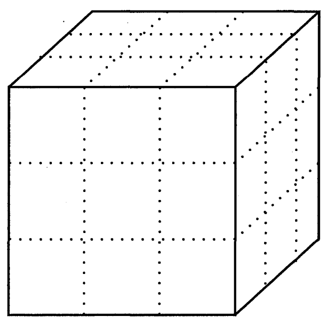
   (A) 12 km/h; (B) $\frac{44}{3}$ km/h; (C) 16 km/h; (D) 17 km/h; (E) 18 km/h.

6. In the diagram, ΔABC and ΔCDE are equilateral triangles. Given that $\angle EBD = 62°$ and $\angle AEB = x°$, what is the value of x?



(A) 100; (B) 118; (C) 120; (D) 122; (E) 135.

7. A carpenter wishes to cut a wooden $3 \times 3 \times 3$ cube into twenty seven $1 \times 1 \times 1$ cubes. He can do this easily by making 6 cuts through the cube, keeping the pieces together in the cube shape as shown:



What is the minimum number of cuts needed if he is allowed to rearrange the pieces after each cut?

(A) 2; (B) 3; (C) 4; (D) 5; (E) 6.

8. What is the last digit of $7^{(7^7)}$?

(A) 1; (B) 3; (C) 5; (D) 7; (E) 9.

9. Given that n is an odd integer less than 1000 and the product of all its digits is 252. How many such integers are there ?

   (A) 3; (B) 4; (C) 5; (D) 6; (E) 7.

10. What is the value of
$$(\sqrt{11}+\sqrt{5})^8+(\sqrt{11}-\sqrt{5})^8?$$

   (A) 451856; (B) 691962; (C) 903712; (D) 1276392; (E) 1576392.

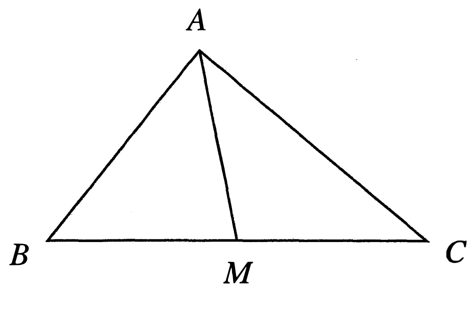
Short Questions

11. Let x and y be real numbers satisfying
$$y=\sqrt{\frac{2008x+2009}{2010x-2011}}+\sqrt{\frac{2008x+2009}{2011-2010x}}+2010.$$
Find the value of y.

12. For integers $a_1,\ldots,a_n\in\{1,2,3,\ldots,9\}$, we use the notation $\overline{a_1a_2\ldots a_n}$ to denote the number $10^{n-1}a_1+10^{n-2}a_2+\cdots+10a_{n-1}+a_n$. For example, when $a=2$ and $b=0$, $\overline{ab}$ denotes the number 20. Given that $\overline{ab}=b^2$ and $\overline{acbc}=(\overline{ba})^2$. Find the value of $\overline{abc}$.

13. Given that $(m-2)$ is a positive integer and it is also a factor of $3m^2-2m+10$. Find the sum of all such values of m.

14. In triangle ABC, $AB=32$ cm, $AC=36$ cm and $BC=44$ cm. If M is the midpoint of BC, find the length of AM in cm.



15. Evaluate

$$\frac{678+690+702+714+\cdots+1998+2010}{3+9+15+21+\cdots+327+333}.$$

16. Esther and Frida are supposed to fill a rectangular array of 16 columns and 10 rows, with the numbers 1 to 160. Esther chose to do it row-wise so that the first row is numbered $1, 2, \ldots, 16$ and the second row is $17, 18, \ldots, 32$ and so on. Frida chose to do it column-wise, so that her first column has $1, 2, \ldots, 10$, and the second column has $11, 12, \ldots, 20$ and so on. Comparing Esther's array with Frida's array, we notice that some numbers occupy the same position. Find the sum of the numbers in these positions.

1	2	3	···	···	16
17	18	19	···	···	32
···	···	···	···	···	···
···	···	···	···	···	···
145	146	147	···	···	160

Esther

1	11	21	···	···	151
2	12	22	···	···	152
···	···	···	···	···	···
···	···	···	···	···	···
10	20	30	···	···	160

Frida

17. The sum of two integers A and B is 2010. If the lowest common multiple of A and B is 14807, write down the larger of the two integers A or B.

18. A sequence of polynomials $a_n(x)$ are defined recursively by

$$\begin{aligned} a_0(x) &= 1, \\ a_1(x) &= x^2 + x + 1, \\ a_n(x) &= (x^n + 1)a_{n-1}(x) - a_{n-2}(x), \text{ for all } n \geq 2. \end{aligned}$$

For example,

$$\begin{aligned} a_2(x) &= (x^2+1)(x^2+x+1) - 1 = x^4 + x^3 + 2x^2 + x, \\ a_3(x) &= (x^3+1)(x^4+x^3+2x^2+x) - (x^2+x+1) \\ &= x^7 + x^6 + 2x^5 + 2x^4 + x^3 + x^2 - 1. \end{aligned}$$

Evaluate $a_{2010}(1)$.

19. A triangle ABC is inscribed in a semicircle of radius 5. If $AB = 10$, find the maximum value of s^2 where $s = AC + BC$.

20. Find the last two digits of $2011^{(2010^{2009})}$.

21. Your national football coach brought a squad of 18 players to the 2010 World Cup, consisting of 3 goalkeepers, 5 defenders, 5 midfielders and 5 strikers. Midfielders are versatile enough to play as both defenders and midfielders, while the other players can only play in their designated positions. How many possible teams of 1 goalkeeper, 4 defenders, 4 midfielders and 2 strikers can the coach field?

22. Given that $169(157-77x)^2+100(201-100x)^2=26(77x-157)(1000x-2010)$, find the value of x.

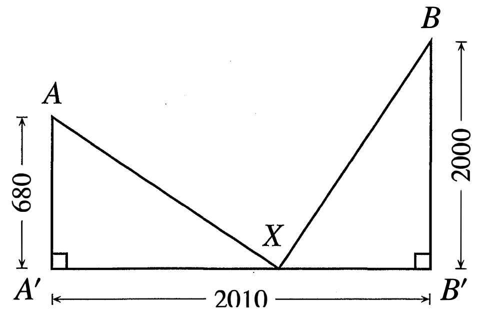
23. Evaluate

$$\frac{(2020^2-20100)(20100^2-100^2)(2000^2+20100)}{2010^6-10^6}.$$

24. When 15 is added to a number x, it becomes a square number. When 74 is subtracted from x, the result is again a square number. Find the number x.

25. Given that x and y are positive integers such that $56 \le x+y \le 59$ and $0.9 < \frac{x}{y} < 0.91$, find the value of $y^2 - x^2$.

26. Let AA' and BB' be two line segments which are perpendicular to $A'B'$. The lengths of AA', BB' and $A'B'$ are 680, 2000 and 2010 respectively. Find the minimal length of $AX + XB$ where X is a point between A' and B'.

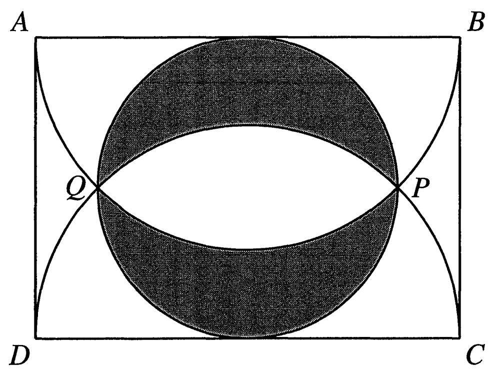


27. The product $1 \times 2 \times 3 \times \cdots \times n$ is denoted by $n!$. For example $4! = 1 \times 2 \times 3 \times 4 = 24$. Let $M = 1! \times 2! \times 3! \times 4! \times 5! \times 6! \times 7! \times 8! \times 9!$. How many factors of M are perfect squares?

28. Starting from any of the L's, the word $LEVEL$ can be spelled by moving either up, down, left or right to an adjacent letter. If the same letter may be used twice in each spell, how many different ways are there to spell the word $LEVEL$?

		L		
	L	E	L	
L	E	V	E	L
	L	E	L	
		L		

29. Let $ABCD$ be a rectangle with $AB = 10$. Draw circles C_1 and C_2 with diameters AB and CD respectively. Let P, Q be the intersection points of C_1 and C_2. If the circle with diameter PQ is tangent to AB and CD, then what is the area of the shaded region?



30. Find the least prime factor of

$$1\underbrace{0000\cdots00}_{2010\text{-many}}1.$$

31. Consider the identity $1+2+\cdots+n = \frac{1}{2}n(n+1)$. If we set $P_1(x) = \frac{1}{2}x(x+1)$, then it is the unique polynomial such that for all positive integer n, $P_1(n) = 1+2+\cdots+n$. In general, for each positive integer k, there is a unique polynomial $P_k(x)$ such that

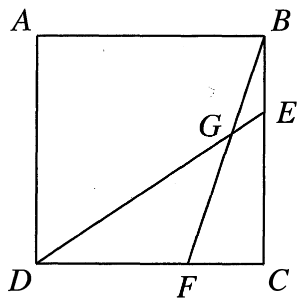
$$P_k(n) = 1^k + 2^k + 3^k + \cdots + n^k \quad \text{for each } n = 1, 2, \ldots.$$

Find the value of $P_{2010}(-\frac{1}{2})$.

32. Given that $ABCD$ is a square. Points E and F lie on the side BC and CD respectively. such that $BE = CF = \frac{1}{3}AB$. G is the intersection of BF and DE. If

$$\frac{\text{Area of } ABGD}{\text{Area of } ABCD} = \frac{m}{n}$$

is in its lowest term, find the value of $m+n$.



33. It is known that there is only one pair of positive integers a and b such that $a \le b$ and $a^2 + b^2 + 8ab = 2010$. Find the value of $a+b$.

34. The digits of the number 123456789 can be rearranged to form a number that is divisible by 11. For example, 123475869, 459267831 and 987453126. How many such numbers are there?

35. Suppose the three sides of a triangular field are all integers, and its area equals the perimeter (in numbers). What is the largest possible area of the field?

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Junior Section Solutions)

1. Ans: (E)

   It is the only number less than 1.

2. Ans: (D)

   Other than (D), all numbers are less than 2010^{2010}. Now $3^7 > 2010$. Thus $2010^{2010} < (3^7)^{3^7} < 3^{(3^9)}$, the result follows.

3. Ans: (B)

   Only the third statement is correct: $a < b$ implies $a + c < b + c$. For other statements, counterexamples can be taken as $a = -1, b = 1; c = 0$ and $a = 0, b = -1$ respectively.

4. Ans: (C)

   Since $(2010 + 1)^3 = 2010^3 + 3 \times 2010^2 + 3 \times 2010 + 1$. The result follows.

5. Ans: (A)

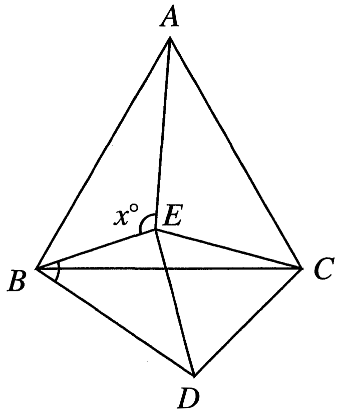
   Let the distance between Town A and Town B be $3s$. The total time taken for up slope, horizontal and down slope road are $\frac{s}{8}$, $\frac{s}{12}$ and $\frac{s}{24}$ respectively. His average speed for the whole journey is $\dfrac{3s}{\frac{s}{8} + \frac{s}{12} + \frac{s}{24}} = 12$ km/h.

6. Ans: (D)

   Observe that ΔBCD is congruent to ΔACE (using SAS, $BC = AC, CD = CE$ and $\angle ACE = 60^\circ - \angle ECB = \angle BCD$). Thus $\angle AEC = \angle BDC$. We get

$$\begin{aligned} x^\circ &= 360^\circ - \angle AEC - \angle BEC \\ &= 360^\circ - \angle BDC - \angle BEC \\ &= \angle EBD + \angle ECD \\ &= 62^\circ + 60^\circ = 122^\circ \end{aligned}$$

   Thus $x = 122$.



7. Ans: (E)

   There is no way to reduce the cuts to fewer than 6: Just consider the middle cube (the one which has no exposed surfaces in the beginning), each of the its sides requires at least one cut.

8. Ans: (B)

   The last digit of 7^k is $1, 7, 9, 3$ respectively for $k \equiv 0, 1, 2, 3 \pmod 4$.
   Since $7^7 \equiv (-1)^7 \equiv 3 \pmod 4$, the last digit of $7^{(7^7)}$ is 3.

9. Ans: (C)

   $252 = 2 \times 2 \times 3 \times 3 \times 7$. We can have: $667, 497, 479, 947, 749$.

10. Ans: (C)

    We only need a rough estimate to rule out the wrong answers. $\sqrt{11} \approx 3.3$ and $\sqrt{5} \approx 2.2$, so the sum is $\approx 5.5^8 + (1.1)^8 \approx 5.5^8 = 30.25^4 \approx 30^4 \approx 810000$. Thus (C). Of course the exact answer can be obtained by calculations, for example,

$$\begin{aligned}
&\sum_{i=0}^{8} \binom{8}{i} (\sqrt{11})^i (\sqrt{5})^{8-i} + \sum_{i=0}^{8} \binom{8}{i} (\sqrt{11})^i (-\sqrt{5})^{8-i} \\
&= 2 \sum_{i \text{ even}}^{8} \binom{8}{i} (\sqrt{11})^i (\sqrt{5})^{8-i} \\
&= 2 \sum_{j=0}^{4} \binom{8}{2j} (11)^j (5)^{4-j}.
\end{aligned}$$

11. Ans: 2010.

Let

$$a = \frac{2008x + 2009}{2010x - 2011}.$$

Then $a \geq 0$ and $-a \geq 0$ since they are under the square root. Hence $a = 0$. Thus $y = 2010$.

12. Ans: 369.

It is easy to see that $b = 5$, $\overline{ab} = 25$ or $b = 6$, $\overline{ab} = 36$. Upon checking, $b = 6$, $a = 3$ and so $(\overline{ba})^2 = 63^2 = 3969$. Therefore $c = 9$. Hence $\overline{abc} = 369$.

13. Ans: 51.

Since

$$\frac{3m^2 - 2m + 10}{m - 2} = 3m + 4 + \frac{18}{m - 2}$$

is an integer. Thus $m - 2$ is a factor of 18. $m - 2 = 1, 2, 3, 6, 9, 18$, thus $m = 3, 4, 5, 8, 11, 20$. The required sum is 51.

14. Ans: 26.

Using the Median Formula, $AM^2 = \dfrac{2AB^2 + 2AC^2 - BC^2}{4}$. Thus $AM = 26$ cm.

15. Ans: 16.

Note that both numerator and denominator are arithmetic progressions. Removing common factors gives

$$2 \times \frac{113 + 115 + 117 + \ldots + 333 + 335}{1 + 3 + 5 + \ldots + 109 + 111}$$

$$= \frac{2 \times \frac{112}{2}(335 + 113)}{\frac{56}{2}(111 + 1)} = 16.$$

16. Ans: 322.

The number in the r-th row and c-th column has value $16(r - 1) + c$ in Esther's array and value $10(c - 1) + r$ in Frida's array. So we need to solve

$$16(r - 1) + c = 10(c - 1) + r \implies 5r = 3c + 2.$$

There are exactly four solutions

$$(r, c) = \{(1, 1), (4, 6), (7, 11), (10, 16)\}.$$

17. Ans: 1139.

A direct way to solve the problem is to factor 14807 directly. Alternatively, one may hope for A and B to have common factors to simplify the problem. This is a good strategy because of the following fact:

> "The greatest common divisor of A and B, equals the greatest common divisor of $A + B$ and $\operatorname{lcm}(A, B)$."

2010 is easily factored as $2 \times 3 \times 5 \times 67$. Checking that 67 is also a factor of 14807, we can conclude that 67 is also a factor of A and B. The problem is reduced to finding a and b such that

$$a + b = \frac{2010}{67} = 30 \qquad \text{and} \qquad ab = \frac{14807}{67} = 221.$$

Since 221 can be factored easily, a and b must be 13 and 17. So the answer is $17 \times 67 = 1139$.

18. Ans: 4021.

$a_n(1)$ is the simple recurrence relation $f_n = 2f_{n-1} - f_{n-2}$, $f_0 = 1$ and $f_1 = 3$. Using standard technique or simply guess and verify that $f_n = 2n + 1$. So $a_{2010}(1) = f_{2010} = 2(2010) + 1$.

19. Ans: 200.

ABC must be a right-angled triangle. Let $x = AC$ and $y = BC$, by Pythagoras theorem $x^2 + y^2 = 10^2$.

$$s^2 = (x + y)^2 = x^2 + y^2 + 2xy = 100 + 2 \times \text{ area of } ABC.$$

Maximum area occurs when $x = y$, i.e. $\angle CAB = 45°$. So $x = y = \sqrt{50}$.

20. Ans: 01.

Note that $2011 \equiv 11 \pmod{100}$ and $11^2 \equiv 21, 11^3 \equiv 31 \pmod{100}$ etc. So $11^{10} \equiv 1 \pmod{100}$. Since 2010^{2009} is divisible by 10,

$$2011^{2010^{2009}} \equiv 11^{10} \times \cdots \times 11^{10} \equiv 1 \pmod{100}.$$

21. Ans: 2250.

$\binom{3}{1} \times \binom{5}{2} \times \binom{5}{4}$ choices for goalkeepers, strikers and midfielders respectively. The remaining midfielder and defenders can all play as defenders, hence total number of possibilities are

$$3 \times 10 \times 5 \times \binom{6}{4} = 2250.$$

22. Ans: 31.

Let $a = 1001x - 2041$ and $b = 1000x - 2010$.

Then the equation becomes $a^2 + b^2 = 2ab$. Thus $(a - b)^2 = 0$. The result follows.

23. Ans: 100.

Let $x = 2010$ and $y = 10$. The numerator becomes

$$\begin{aligned} & [(x+y)^2 - xy] \cdot (x^2y^2 - y^2y^2) \cdot [(x-y)^2 + xy] \\ = \; & (x^2 + xy + y^2) \cdot y^2(x-y)(x+y) \cdot (x^2 - xy + y^2) \\ = \; & y^2(x^3 - y^3)(x^3 + y^3) \\ = \; & y^2(x^6 - y^6). \end{aligned}$$

Hence the answer is 100.

24. Ans: 2010.

Let $15 + x = m^2$ and $x - 74 = n^2$. We have $m^2 - n^2 = 89 = 1 \times 89$. $(m-n)(m+n) = 1 \times 89$. Let $m - n = 1$ and $m + n = 89$. Solving $m = 45$ and $n = 44$. Thus the number x is $45^2 - 15 = 2010$.

25. Ans: 177.

From $0.9y < x < 0.91y$, we get $0.9y + y < x + y < 0.91y + y$. Thus $0.9y + y < 59$ and $0.91y + y > 56$. It follows that $y < 31.05$ and $y > 29.3$. Thus $y = 30$ or 31. If $y = 30$, then $27 < x < 27.3$, no integer value of x. If $y = 31$, then $27.9 < x < 28.21$, thus $x = 28$. Thus $y^2 - x^2 = (31 + 28)(31 - 28) = 177$.

26. Ans: 3350.

Take the reflection with respect to $A'B'$. Let A'' be the image of A. Then the minimal length is equal to the length of $A''B = \sqrt{2010^2 + (2000 + 680)^2} = 3350$.

27. Ans: 672.

$$\begin{aligned} M &= 1! \times 2! \times 3! \times 4! \times 5! \times 6! \times 7! \times 8! \times 9! \\ &= 2^8 \times 3^7 \times 4^6 \times 5^5 \times 6^4 \times 7^3 \times 8^2 \times 9 \\ &= 2^{30} \times 3^{13} \times 5^5 \times 7^3 \end{aligned}$$

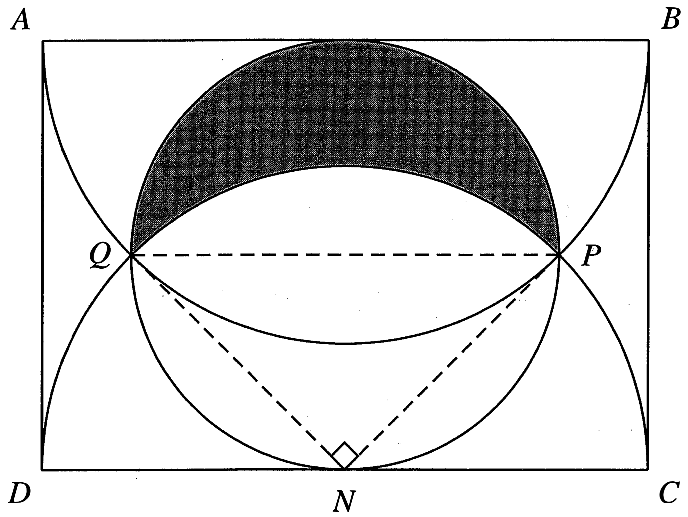
A perfect square factor of M must be of the form $2^{2x} \times 3^{2y} \times 5^{2z} \times 7^{2w}$, where x, y, z and w are whole numbers such that $2x \leq 30$, $2y \leq 13$, $2z \leq 5$, $2w \leq 3$. Hence, the number of perfect square factors of M is $16 \times 7 \times 3 \times 2 = 672$.

28. Ans: 144.

There are total 12 ways starting from any of the L's to reach the middle V. Hence the total number of ways to spell the word *LEVEL* is $12^2 = 144$.

29. Ans: 25.

Let N be the midpoint of CD. Then $\angle PNQ = 90°$. So $PQ = 5\sqrt{2}$.



Then the area of the shaded region is

$$A = 2\left[\frac{1}{2}\pi\left(\frac{PQ}{2}\right)^2 + \frac{1}{2}(PN)^2 - \frac{1}{4}\pi(PN)^2\right]$$
$$= 2\left[\frac{1}{2}\pi\left(\frac{5\sqrt{2}}{2}\right)^2 + \frac{1}{2}\cdot 5^2 - \frac{1}{4}\pi\cdot 5^2\right] = 25.$$

30. Ans: 11.

Clearly 11 is a factor and 2, 3, 5 are not. We only need to rule out 7. $10^{2011} + 1 \equiv 4 \pmod 7$ because $10^3 \equiv -1 \pmod 7$.

31. Ans: 0.

Let k be a positive even number.

Define $f(x) = P_k(x) - P_k(x-1)$. Then $f(n) = n^k$ for all integer $n \geq 2$. Note that f is a polynomial. We must have $f(x) = x^k$. In particular, for integers $n \geq 2$,

$$\begin{aligned} P_k(-n+1) - P_k(-n) &= f(-n+1) = (n-1)^k, \\ P_k(-n+2) - P_k(-n+1) &= f(-n+2) = (n-2)^k, \\ &\cdots\cdots \\ P_k(0) - P_k(-1) &= f(0) = 0^k, \\ P_k(1) - P_k(0) &= f(1) = 1^k. \end{aligned}$$

Summing these equalities, $P_k(1) - P_k(-n) = 1^k + 0^k + 1^k + \cdots + (n-1)^k$. That is,

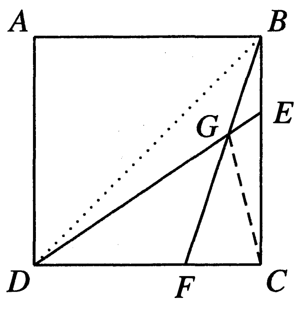
$$P_k(-n) + P_k(n-1) = 0.$$

Define $g(x) = P_k(-x) + P_k(x-1)$. Then $g(n) = 0$ for all integers $n \geq 2$. Since g is a polynomial, $g(x) = 0$.

In particular, $P_k(-\frac{1}{2}) + P_k(-\frac{1}{2}) = 0$, i.e., $P_k(-\frac{1}{2}) = 0$.

32. Ans: 23.

Join BD and CG and note that $\dfrac{DF}{FC} = 2$.



Assume the length of AB is 1. Let the area of ΔBGE and ΔFGC be x and y respectively. Then the areas of ΔEGC and ΔDGF are $2x$ and $2y$. Since the area of ΔBFC is $\frac{1}{3}$, we have $3x + y = \frac{1}{6}$. Similarly, $3y + 2x =$ the area of $\Delta DEC = \frac{1}{3}$. Solve $x = \frac{1}{42}$ and $y = \frac{2}{21}$. Thus

$$\frac{\text{Area of } ABGD}{\text{Area of } ABCD} = 1 - 3(x+y) = 1 - \frac{15}{42} = \frac{9}{14}.$$

So $m = 9$ and $n = 14$. The result follows.

33. Ans: 42.

Since $a \geq 1, 2010 = a^2+b^2+8ab \geq 1+b^2+8b$. $b^2+8b-2009 \leq 0$. However $b^2+8b-2009 = 0$ has an integer solution 41. So $a = 1$ and $b = 41$. The result follows.

34. Ans: 31680.

Let X and Y be the sum of the digits at even and odd positions respectively. Note that $1+2+3+\cdots+9 = 45$. We have $X + Y = 45$ and 11 divides $|X - Y|$. It's easy to see $X = 17$ and $Y = 28$; or $X = 28$ and $Y = 17$. Hence we split the digits into 2 sets whose sum is 17 and 28 respectively.

There are 9 ways for 4 digits to sum to 17: $\{9,5,2,1\}$, $\{9,4,3,1\}$, $\{8,6,2,1\}$, $\{8,5,3,1\}$, $\{8,4,3,2\}$, $\{7,6,3,1\}$, $\{7,5,4,1\}$, $\{7,5,3,2\}$, $\{6,5,4,2\}$. There are 2 ways for 4 digits to sum to 28: $\{9,8,7,4\}$, $\{9,8,6,5\}$. Thus the total number of ways is $11 \times 4! \times 5! = 31680$.

35. Ans. 60.

Let the three sides of the triangle be a, b, c respectively. Then

$$\sqrt{s(s-a)(s-b)(s-c)} = a+b+c = 2s,$$

where $s = \dfrac{a+b+c}{2}$. Note that s is an integer; otherwise $s(s-a)(s-b)(s-a)$ is a non-integer. Let $x = s-a$, $y = s-b$ and $z = s-c$. Then x, y, z are positive integers satisfying

$$xyz = 4(x+y+z).$$

Assume that $x \geq y \geq z$. Then $xyz \leq 12x$, i.e., $yz \leq 12$, and thus $z \leq 3$.

If $z = 1$, $xy = 4(x+y+1)$ implies $(x-4)(y-4) = 20 = 20 \cdot 1 = 10 \cdot 2 = 5 \cdot 4$. So $(x, y) = (24, 5), (14, 6), (9, 8)$.

If $z = 2$, $2xy = 4(x+y+2)$ implies $(x-2)(y-2) = 8 = 8 \cdot 1 = 4 \cdot 2$. So $(x, y) = (10, 3), (6, 4)$.

If $z = 3$, $3xy = 4(x+y+3)$ implies $(3x-4)(3y-4) = 52$, which has no solution $x \geq y \geq 3$.

The area is 60, 42, 36, 30, 24, respectively; and the largest possible value is 60.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Junior Section, Round 2)

Saturday, 25 June 2010 **0930-1230**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

---

1. Let the diagonals of the square $ABCD$ intersect at S and let P be the midpoint of AB. Let M be the intersection of AC and PD and N the intersection of BD and PC. A circle is incribed in the quadrilateral $PMSN$. Prove that the radius of the circle is $MP - MS$.

2. Find the sum of all the 5-digit integers which are not multiples of 11 and whose digits are 1, 3, 4, 7, 9.

3. Let $a_1, a_2, \ldots, a_n$ be positive integers, not necessarily distinct but with at least five distinct values. Suppose that for any $1 \leq i < j \leq n$, there exist k, ℓ, both different from i and j such that $a_i + a_j = a_k + a_\ell$. What is the smallest possible value of n?

4. A student divides an integer m by a positive integer n, where $n \leq 100$, and claims that

$$\frac{m}{n} = 0 \cdot 167 a_1 a_2 \cdots.$$

Show the student must be wrong.

5. The numbers $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2010}$ are written on a blackboard. A student chooses any two of the numbers, say x, y, erases them and then writes down $x + y + xy$. He continues to do this until only one number is left on the blackboard. What is this number?

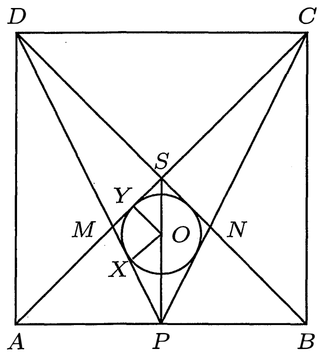
Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Junior Section, Round 2 solutions)

1. Let O be the centre and r the radius of the circle. Let X, Y be its points of contact with the sides PM, MS, respectively.

Since $OY \perp MS$ and $\angle YSO = \angle ASP = 45°$, $SY = YO = r$. Also $\angle OPX = \angle PDA$ (since $OP \parallel DA$) and $\angle OXP = \angle PAD = 90°$. Therefore $\triangle OXP \simeq \triangle PAD$. Hence $OX/XP = PA/AD = 1/2$. Hence $PX = 2r$. Therefore $PM - MS = 2r + MX - MY - r = r$.



2. First note that an integer is divisible by 11 if and only if the alternating sum of the digits is divisible by 11. In our case, these are the integers where 1,4 and 7 are at the odd positions. Let S be the sum of all the 5-digit integers formed by 1, 3, 4, 7, 9 and let T be the sum of those which are multiples of 11. Then

$$\begin{aligned} S &= 4!(1+3+4+7+9)(1+10+100+1000+10000) \\ &= 6399936 \\ T &= 2!2!(1+4+7)(1+100+10000) + 3!(3+9)(10+1000) = 557568. \end{aligned}$$

Thus the sum is $6399936 - 557568 = 5842368$.

3. $a_1 \le a_2 \le \cdots \le a_n$. Suppose $x < y$ are the two smallest values. Then $a_1 = x$ and let s be the smallest index such that $a_s = y$. Now there are two other terms whose sum is $x + y$. Thus we have $a_2 = x$ and $a_{s+1} = y$. Since $a_1 + a_2 = 2x$, we must have $a_3 = a_4 = x$. Similarly, by considering the largest two values $w < z$, we have $a_n = a_{n-1} = a_{n-2} = a_{n-3} = z$ and another two terms equal to w. Since there is one other value, there are at least $4 + 2 + 4 + 2 + 1 = 13$ terms. The following 13 numbers satisfy the required property: $1, 1, 1, 1, 2, 2, 3, 4, 4, 5, 5, 5, 5$. Thus the smallest possible value of n is 13.

4. We have

$$0 \cdot 167 \leq \frac{m}{n} < 0 \cdot 168 \quad \Rightarrow \quad 167n \leq 1000m < 168n.$$

Multiply by 6, we get

$$1002n \leq 6000m < 1008n \quad \Rightarrow \quad 6000m - 1000n < 8n \leq 800.$$

But $6000m - 1000n \geq 2n > 0$. Thus $6000m - 1000n \geq 1000$ since it is a multiple of 1000. We thus get a contradiction.

5. We shall prove by induction that if the original numbers are $a_1, \ldots, a_n$, $n \geq 2$, then the last number is $(1 + a_1) \cdots (1 + a_n) - 1$.

The assertion is certainly true for $n = 2$, the base case. Now suppose it is true for $n = k \geq 2$. Consider $k + 1$ numbers $a_1, \ldots, a_{k+1}$ written on the board. After one operation, we are left with k numbers. Without loss of generality, we can assume that the student erases a_k and a_{k+1} and writes $b_k = a_k + a_{k+1} + a_k a_{k+1} = (1 + a_k)(1 + a_{k+1}) - 1$. After a further k operations, we are left with the number

$$(1 + a_1) \cdots (1 + a_{k-1})(1 + b_k) - 1 = (1 + a_1) \cdots (1 + a_{k-1})(1 + a_k)(1 + a_{k+1}) - 1.$$

This completes the proof of the inductive step. Thus the last number is

$$\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{2010}\right) - 1 = 2010$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Senior Section)

Tuesday, 1 June 2010 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D *or* E) *corresponding to the correct answer.*

For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Find the value of $\dfrac{(1\times 2\times 3)+(2\times 4\times 6)+(3\times 6\times 9)+\cdots+(335\times 670\times 1005)}{(1\times 3\times 6)+(2\times 6\times 12)+(3\times 9\times 18)+\cdots+(335\times 1005\times 2010)}$.

(A) $\frac{1}{3}$

(B) $\frac{2}{3}$

(C) $\frac{1}{6}$

(D) $\frac{1}{2}$

(E) $\frac{4}{9}$

2. If a, b, c and d are real numbers such that

$$\frac{b+c+d}{a}=\frac{a+c+d}{b}=\frac{a+b+d}{c}=\frac{a+b+c}{d}=r,$$

find the value of r.

(A) 3
(B) 1
(C) –1
(D) 3 or 1
(E) 3 or –1

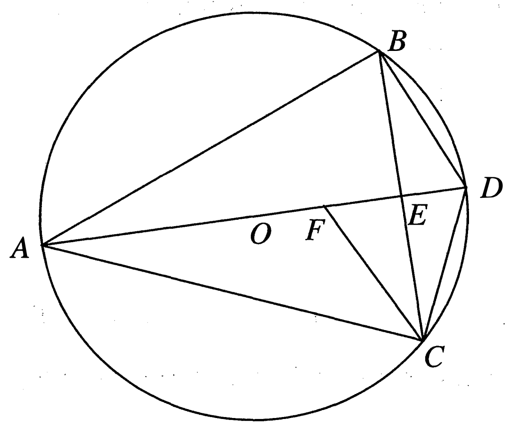
3. If $0<x<\frac{\pi}{2}$ and $\sin x-\cos x=\frac{\pi}{4}$ and $\tan x+\frac{1}{\tan x}=\frac{a}{b-\pi^c}$, where a, b and c are positive integers, find the value of $a+b+c$.

(A) 8
(B) 32
(C) 34
(D) 48
(E) 50

4. Find the value of $\sqrt{14^3+15^3+16^3+\cdots+24^3+25^3}$.

(A) 104
(B) 224
(C) 312
(D) 336
(E) 676

5. In the figure below, ABC is an isosceles triangle inscribed in a circle with centre O and diameter AD, with $AB = AC$. AD intersects BC at E, and F is the midpoint of OE. Given that BD is parallel to FC and $BC = 2\sqrt{5}$ cm, find the length of CD in cm.



(A) $\frac{3\sqrt{5}}{2}$
(B) $\sqrt{6}$
(C) $2\sqrt{3}$
(D) $\sqrt{7}$
(E) $2\sqrt{6}$

6. Find the number of ordered pairs (x, y), where x is an integer and y is a perfect square, such that $y = (x - 90)^2 - 4907$.

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4

7. Let $S = \{1, 2, 3, \ldots, 9, 10\}$. A non-empty subset of S is considered "Good" if the number of even integers in the subset is more than or equal to the number of odd integers in the same subset. For example, the subsets $\{4, 8\}$, $\{3, 4, 7, 8\}$ and $\{1, 3, 6, 8, 10\}$ are "Good". How many subsets of S are "Good"?

(A) 482
(B) 507
(C) 575
(D) 637
(E) 667

8. If the graph of a quadratic function $f(x) = ax^2 + bx + c$ ($a \neq 0$) passes through two distinct points (r, k) and (s, k), what is $f(r+s)$?

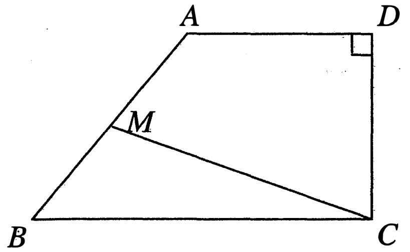
(A) $2k$
(B) c
(C) $k - c$
(D) $2k - c$
(E) None of the above

9. Find the number of positive integers $k < 100$ such that $2(3^{6n}) + k(2^{3n+1}) - 1$ is divisible by 7 for any positive integer n.

(A) 10
(B) 12
(C) 13
(D) 14
(E) 16

10. Let $ABCD$ be a trapezium with AD parallel to BC and $\angle ADC = 90^\circ$, as shown in the figure below. Given that M is the midpoint of AB with $CM = \frac{13}{2}$ cm and $BC + CD + DA = 17$ cm, find the area of the trapezium $ABCD$ in cm^2.

(A) 26
(B) 28
(C) 30
(D) 33
(E) 35



Short Questions

11. The area of a rectangle remains unchanged when either its length is increased by 6 units and width decreased by 2 units, or its length decreased by 12 units and its width increased by 6 units. If the perimeter of the original rectangle is x units, find the value of x.

12. For $r = 1, 2, 3, \ldots$, let $u_r = 1+2+3+\ldots+r$. Find the value of

$$\frac{1}{\left(\frac{1}{u_1}\right)}+\frac{2}{\left(\frac{1}{u_1}+\frac{1}{u_2}\right)}+\frac{3}{\left(\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right)}+\cdots+\frac{100}{\left(\frac{1}{u_1}+\frac{1}{u_2}+\cdots+\frac{1}{u_{100}}\right)}.$$

13. If $2010! = M\times 10^k$, where M is an integer not divisible by 10, find the value of k.

14. If $a>b>1$ and $\frac{1}{\log_a b}+\frac{1}{\log_b a}=\sqrt{1229}$, find the value of $\frac{1}{\log_{ab} b}-\frac{1}{\log_{ab} a}$.

15. For any real number x, let $\lceil x \rceil$ denote the smallest integer that is greater than or equal to x and $\lfloor x \rfloor$ denote the largest integer that is less than or equal to x (for example, $\lceil 1.23 \rceil = 2$ and $\lfloor 1.23 \rfloor = 1$). Find the value of

$$\sum_{k=1}^{2010}\left[\frac{2010}{k}-\left\lfloor\frac{2010}{k}\right\rfloor\right].$$

16. Let $f(x)=\frac{x^{2010}}{x^{2010}+(1-x)^{2010}}$. Find the value of

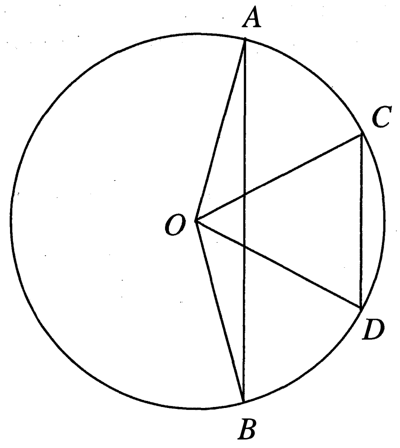
$$f\left(\frac{1}{2011}\right)+f\left(\frac{2}{2011}\right)+f\left(\frac{3}{2011}\right)+\ldots+f\left(\frac{2010}{2011}\right).$$

17. If a, b and c are positive real numbers such that

$$ab + a + b = bc + b + c = ca + c + a = 35,$$

find the value of $(a + 1)(b + 1)(c + 1)$.

18. In the figure below, AB and CD are parallel chords of a circle with centre O and radius r cm. It is given that $AB = 46$ cm, $CD = 18$ cm and $\angle AOB = 3 \times \angle COD$. Find the value of r.



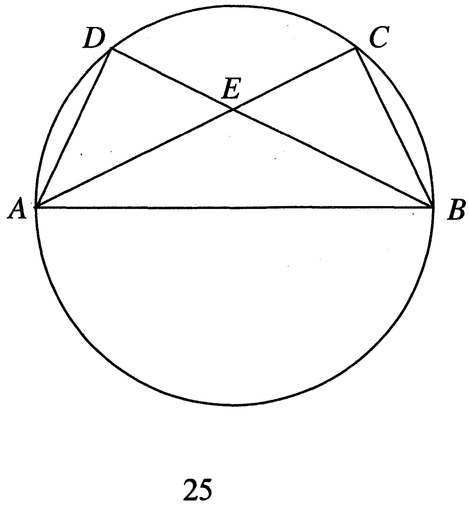
19. Find the number of ways that 2010 can be written as a sum of one or more positive integers in non-decreasing order such that the difference between the last term and the first term is at most 1.

20. Find the largest possible value of n such that there exist n consecutive positive integers whose sum is equal to 2010.

21. Determine the number of pairs of positive integers n and m such that

$$1! + 2! + 3! + \cdots + n! = m^2.$$

22. The figure below shows a circle with diameter AB. C and D are points on the circle on the same side of AB such that BD bisects $\angle CBA$. The chords AC and BD intersect at E. It is given that $AE = 169$ cm and $EC = 119$ cm. If $ED = x$ cm, find the value of x.



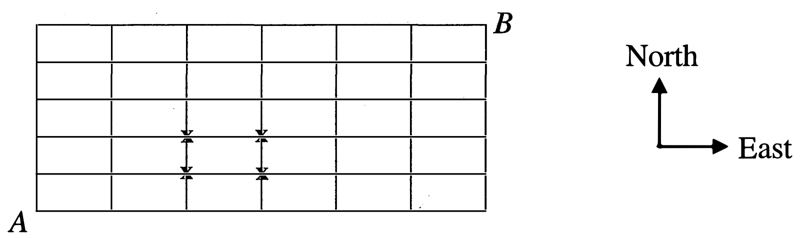
23. Find the number of ordered pairs (m, n) of positive integers m and n such that $m + n = 190$ and m and n are relatively prime.

24. Find the least possible value of $f(x)=\frac{9}{1+\cos 2x}+\frac{25}{1-\cos 2x}$, where x ranges over all real numbers for which $f(x)$ is defined.

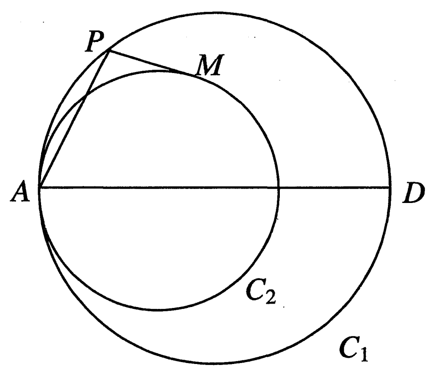
25. Find the number of ways of arranging 13 identical blue balls and 5 identical red balls on a straight line such that between any 2 red balls there is at least 1 blue ball.

26. Let $S = \{1, 2, 3, 4, \ldots, 100000\}$. Find the least possible value of k such that any subset A of S with $|A| = 2010$ contains two distinct numbers a and b with $|a - b| \leq k$.

27. Find the number of ways of traveling from A to B, as shown in the figure below, if you are only allowed to walk east or north along the grid, and avoiding all the 4 points marked x.



28. Two circles C_1 and C_2 of radii 10 cm and 8 cm respectively are tangent to each other internally at a point A. AD is the diameter of C_1 and P and M are points on C_1 and C_2 respectively such that PM is tangent to C_2, as shown in the figure below. If $PM = \sqrt{20}$ cm and $\angle PAD = x^\circ$, find the value of x.



29. Let a, b and c be integers with $a > b > c > 0$. If b and c are relatively prime, $b + c$ is a multiple of a, and $a + c$ is a multiple of b, determine the value of abc.

30. Find the number of subsets $\{a, b, c\}$ of $\{1, 2, 3, 4, \ldots, 20\}$ such that $a < b - 1 < c - 3$.

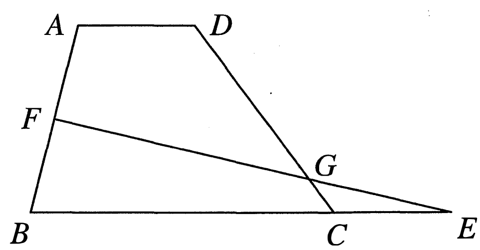
31. Let $f(n)$ denote the number of 0's in the decimal representation of the positive integer n. For example, $f(10001123) = 3$ and $f(1234567) = 0$. Let

$$M = f(1)\times 2^{f(1)} + f(2)\times 2^{f(2)} + f(3)\times 2^{f(3)} + \cdots + f(99999)\times 2^{f(99999)}.$$

Find the value of $M - 100000$.

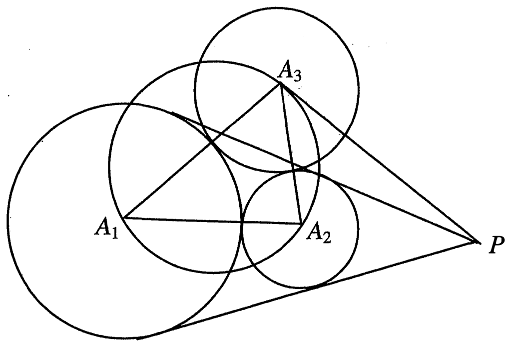
32. Determine the odd prime number p such that the sum of digits of the number $p^4 - 5p^2 + 13$ is the smallest possible.

33. The figure below shows a trapezium $ABCD$ in which $AD \parallel BC$ and $BC = 3AD$. F is the midpoint of AB and E lies on BC extended so that $BC = 3CE$. The line segments EF and CD meet at the point G. It is given that the area of triangle GCE is 15 cm^2 and the area of trapezium $ABCD$ is k cm^2. Find the value of k.



34. Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial in x where the coefficients $a_0, a_1, a_2, \ldots, a_n$ are non-negative integers. If $P(1) = 25$ and $P(27) = 1771769$, find the value of $a_0 + 2a_1 + 3a_2 + \cdots + (n+1)a_n$.

35. Let three circles Γ_1, Γ_2, Γ_3 with centres A_1, A_2, A_3 and radii r_1, r_2, r_3 respectively be mutually tangent to each other externally. Suppose that the tangent to the circumcircle of the triangle $A_1A_2A_3$ at A_3 and the two external common tangents of Γ_1 and Γ_2 meet at a common point P, as shown in the figure below. Given that $r_1 = 18$ cm, $r_2 = 8$ cm and $r_3 = k$ cm, find the value of k.



Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Senior Section Solutions)

1. Answer: (A)

$$\frac{(1\times 2\times 3)+(2\times 4\times 6)+(3\times 6\times 9)+\cdots+(335\times 670\times 1005)}{(1\times 3\times 6)+(2\times 6\times 12)+(3\times 9\times 18)+\cdots+(335\times 1005\times 2010)}$$

$$=\frac{(1\times 2\times 3)[1^3+2^3+3^3+\cdots+335^3]}{(1\times 3\times 6)[1^3+2^3+3^3+\cdots+335^3]}$$

$$=\frac{1\times 2\times 3}{1\times 3\times 6}=\frac{1}{3}.$$

2. Answer: (E)

From the given equations, we obtain

$$a+b+c+d=a(r+1),\quad a+b+c+d=b(r+1),$$
$$a+b+c+d=c(r+1),\quad a+b+c+d=d(r+1).$$

Adding these four equations gives

$$4(a+b+c+d)=(a+b+c+d)(r+1),$$

that is,

$$(3-r)(a+b+c+d)=0.$$

Thus $r = 3$, or $a+b+c+d=0$. If $a+b+c+d=0$, then we see from the original given equations that $r = -1$. Hence the value of r is either 3 or -1.

3. Answer: (E)

We have $(\sin x-\cos x)^2=\frac{\pi^2}{16}$, which implies that $\sin x\cos x=\frac{16-\pi^2}{32}$.

Therefore we obtain

$$\tan x+\frac{1}{\tan x}=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=\frac{1}{\sin x\cos x}=\frac{32}{16-\pi^2}.$$

Hence $a + b + c = 32 + 16 + 2 = 50$.

4. Answer: (C)

First, we note that $4n^3 = [n(n+1)]^2-[n(n-1)]^2$. Thus

$$n^3 = \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{n(n-1)}{2}\right)^2.$$

Therefore

$$\begin{aligned}
&14^3 + 15^3 + \ldots + 24^3 + 25^3 \\
&= \frac{14^2 \times 15^2}{4} - \frac{13^2 \times 14^2}{4} + \\
&\quad \frac{15^2 \times 16^2}{4} - \frac{14^2 \times 15^2}{4} + \ldots. + \\
&\quad \frac{24^2 \times 25^2}{4} - \frac{23^2 \times 24^2}{4} + \\
&\quad \frac{25^2 \times 26^2}{4} - \frac{24^2 \times 25^2}{4} \\
&= \frac{25^2 \times 26^2}{4} - \frac{13^2 \times 14^2}{4} = (25 \times 13 + 13 \times 7)(25 \times 13 - 13 \times 7) \\
&= (32 \times 13)(18 \times 13) = 9 \times 64 \times 13^2.
\end{aligned}$$

Thus $\sqrt{14^3 + 15^3 + 16^3 + \ldots + 24^3 + 25^3} = 3 \times 8 \times 13 = 312.$

5. Answer: (B)

Since the diameter AD perpendicularly bisects the chord BC,

$$BE = EC = \sqrt{5}.$$

Also, given that $BD // FC$, we have $\angle DBE = \angle FCE$. Thus ΔBDE is congruent to ΔCFE, so $DE = FE$. As F is the midpoint of OE, we have $OF = FE = ED$.
Let $OF = x$. Then $AE = 5x$.
Using Intersection Chord Theorem, we have

$$AE \times ED = BE \times EC,$$

which leads to $5x^2 = 5$. Consequently we obtain $x = 1$. Now $CD^2 = CE^2 + ED^2$ gives $CD = \sqrt{5+1} = \sqrt{6}$.

6. Answer: (E)

Let $y = m^2$ and $(x - 90)^2 = k^2$, where m and k are positive integers. Then we obtain $k^2 - m^2 = 4907 = 7 \times 701 = 1 \times 4907$, which gives

$$(k-m)(k+m) = 7 \times 701 \quad \text{or} \quad (k-m)(k+m) = 1 \times 4907.$$

It follows that

$$k - m = 7 \text{ and } k + m = 701, \quad \text{or} \quad k - m = 1 \text{ and } k + m = 4907.$$

Solving these two pairs of equations gives

$$(k, m) = (354, 347) \text{ and } (k, m) = (2454, 2453).$$

Therefore the ordered pairs (x, y) that satisfy the given equation are:

$(444, 347^2)$, $(-264, 347^2)$, $(2544, 2453^2)$, $(-2364, 2453^2)$.

Hence the answer is 4.

7. Answer: (D)

Let the number of even integers in a "Good" subset of S be i, where $i = 1, 2, 3, 4, 5$, and the number of odd integers in that subset be j, where $j = 0, 1, 2, \ldots, i$. Then the number of "Good" subsets of S is

$$\sum_{i=1}^{5}\left(\binom{5}{i}\sum_{j=0}^{i}\binom{5}{j}\right) = \binom{5}{1}\left(\binom{5}{0}+\binom{5}{1}\right)+\binom{5}{2}\left(\binom{5}{0}+\binom{5}{1}+\binom{5}{2}\right)+\cdots$$
$$+\binom{5}{5}\left(\binom{5}{0}+\binom{5}{1}+\cdots+\binom{5}{5}\right)$$
$$= 5(1+5)+10(1+5+10)+\cdots+(1+5+10+10+5+1)$$
$$= 30+160+260+155+32 = 637.$$

8. Answer: (B)

Let $g(x) = f(x) - k$. Then $g(r) = f(r) - k = k - k = 0$. Similarly, $g(s) = 0$. Therefore r and s are roots of the quadratic equation $g(x) = ax^2 + bx + c - k = 0$, from which we deduce that $r + s = -\frac{b}{a}$. Hence

$$f(r+s) = f\left(-\frac{b}{a}\right) = a\left(-\frac{b}{a}\right)^2 + b\left(-\frac{b}{a}\right) + c = c.$$

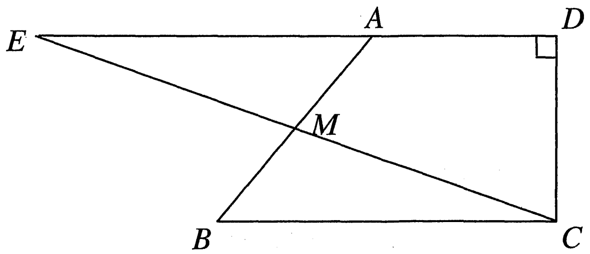
9. Answer: (D)

We have

$$2(3^{6n}) + k(2^{3n+1}) - 1 = 2(27^{2n}) + 2k(8^n) - 1 \equiv 2(-1)^{2n} + 2k(1^n) - 1 \pmod 7$$
$$\equiv 2k + 1 \pmod 7.$$

Thus, for any positive integer n, $2(3^{6n}) + k(2^{3n+1}) - 1$ is divisible by 7 if and only if $2k + 1 \equiv 0 \pmod 7$. As $k < 100$, it is clear that the congruence holds for $k = 3, 10, 17, \ldots, 94$. Thus the required number of positive integers k is 14.

10. Answer: (C)



Extend DA and CM to meet at E as shown in the figure above. Since $AM = MB$, $\angle AEM = \angle BCM$ and $\angle AME = \angle BMC$, we conclude that ΔAEM is congruent to ΔBCM. Therefore $AE = BC$ and $CM = EM$. Thus $CE = 2CM = 13$.

Let the area of trapezium $ABCD$ be S cm^2. Then $S = \frac{1}{2}(DE)(DC)$, and we have

$$(DE+DC)^2 = DE^2 + DC^2 + 2(DE)(DC) = CE^2 + 4S = 13^2 + 4S = 169 + 4S.$$

Now $DE + DC = DA + AE + DC = DA + BC + DC = 17$. Hence $17^2 = 169 + 4S$, and it follows that $S = 30$.

11. Answer: 132

Let the length and width of the original rectangle be L and W respectively. Then

$$LW = (L+6)(W-2) \quad \text{and} \quad LW = (L-12)(W+6).$$

Simplifying the above equations, we obtain

$$L - 2W = 12 \quad \text{and} \quad 3W - L = 6.$$

Solving the simultaneous equations, we get $L = 48$ and $W = 18$. Hence the perimeter of the rectangle is 132 units.

12. Answer: 2575

As $u_r = 1+2+3+...+r = \frac{r(r+1)}{2}$, we have

$$\sum_{r=1}^{i} \frac{1}{u_r} = \sum_{r=1}^{i} \frac{2}{r(r+1)} = \sum_{r=1}^{i} \left(\frac{2}{r} - \frac{2}{r+1}\right) = 2 - \frac{2}{i+1} = \frac{2i}{i+1}.$$

Hence,

$$S_n := \sum_{i=1}^{n} \frac{i}{\sum_{r=1}^{i} \frac{1}{u_r}} = \sum_{i=1}^{n} \left(\frac{i}{\frac{2i}{i+1}}\right) = \sum_{i=1}^{n} \left(\frac{i+1}{2}\right) = \frac{1}{2}\left(\frac{n(n+1)}{2} + n\right) = \frac{n}{4}(n+3).$$

In particular, $S_{100} = 2575$.

13. Answer: 501

The number k is the number of the factor 10 that occurs in 2010!. This number is given by the number of pairs of prime factors 2 and 5 in 2010!.

Now between 1 and 2010, there are:

- 402 integers with 5 as a factor;
- 80 integers with 25 as a factor;
- 16 integers with 125 as a factor;
- 3 integers with 625 as a factor.

Therefore the total number of prime factor 5 in 2010! is 402 + 80 + 16 + 3 = 501. As there are clearly more than 501 prime factor 2 in 2010!, we obtain k = 501.

14. Answer: 35

First note that since $a > b > 1$, $\dfrac{1}{\log_{ab} b} - \dfrac{1}{\log_{ab} a} > 0$. Then

$$\begin{aligned}
\frac{1}{\log_{ab} b} - \frac{1}{\log_{ab} a} &= \log_b ab - \log_a ab \\
&= (\log_b a + 1) - (\log_a b + 1) \\
&= \log_b a - \log_a b \\
&= \frac{1}{\log_a b} - \frac{1}{\log_b a} \\
&= \sqrt{\left(\frac{1}{\log_a b} - \frac{1}{\log_b a}\right)^2} \\
&= \sqrt{\left(\frac{1}{\log_a b} + \frac{1}{\log_b a}\right)^2 - 4\left(\frac{1}{\log_a b}\right)\left(\frac{1}{\log_b a}\right)} \\
&= \sqrt{1229 - 4} \\
&= \sqrt{1225} \\
&= 35.
\end{aligned}$$

15. Answer: 1994

Consider k = 1, 2, ..., 2010. If $k \mid 2010$, then $x := \dfrac{2010}{k} - \left\lfloor \dfrac{2010}{k} \right\rfloor = 0$, so $\lceil x \rceil = 0$. If $k \nmid 2010$, then $0 < y := \dfrac{2010}{k} - \left\lfloor \dfrac{2010}{k} \right\rfloor < 1$, so $\lceil y \rceil = 1$.

Since the prime factorization of 2010 is $2 \times 3 \times 5 \times 67$, we see that 2010 has 16 distinct divisors. Hence

$$\begin{aligned}
\sum_{k=1}^{2010} \left\lceil \frac{2010}{k} - \left\lfloor \frac{2010}{k} \right\rfloor \right\rceil &= \sum_{k \mid 2010} \lceil x \rceil + \sum_{k \nmid 2010} \lceil y \rceil \\
&= \text{number of non-divisor of 2010 among } k \\
&= 2010 - 16 = 1994.
\end{aligned}$$

16. Answer: 1005

Observe that $f(x) + f(1-x) = \dfrac{x^{2010}}{x^{2010} + (1-x)^{2010}} + \dfrac{(1-x)^{2010}}{(1-x)^{2010} + x^{2010}} = 1$.

It follows that

$$f\left(\frac{1}{2011}\right)+f\left(\frac{2}{2011}\right)+f\left(\frac{3}{2011}\right)+\ldots+f\left(\frac{2010}{2011}\right)$$
$$=\left\{f\left(\frac{1}{2011}\right)+f\left(\frac{2010}{2011}\right)\right\}+\left\{f\left(\frac{2}{2011}\right)+f\left(\frac{2009}{2011}\right)\right\}+\ldots$$
$$+\left\{f\left(\frac{1005}{2011}\right)+f\left(\frac{1006}{2011}\right)\right\}$$
$= 1005$.

17. Answer: 216

Adding 1 to both sides of the given equation $ab + a + b = 35$, we obtain

$$(a + 1)(b + 1) = 36.$$

Likewise, adding 1 to the other two given equations gives

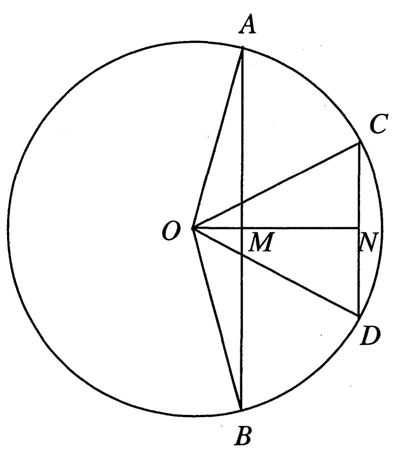
$$(b + 1)(c + 1) = 36 \text{ and } (c + 1)(a + 1) = 36.$$

Now multiplying the three resulting equations above leads to

$$[(a+1)(b+1)(c+1)]^2 = 36^3 = 6^6.$$

It follows that $(a+1)(b+1)(c+1) = 6^3 = 216$.

18. Answer: 27



Let M and N be the midpoints of AB and CD respectively and let $\angle CON = x$. Then $\angle AON = 3x$ and

$$\frac{23}{9}=\frac{AM}{CN}=\frac{r\sin 3x}{r\sin x}=\frac{3\sin x-4\sin^3 x}{\sin x}=3-4\sin^2 x.$$

Thus $\sin^2 x=\frac{1}{4}\left(3-\frac{23}{9}\right)=\frac{1}{9}$, and so $\sin x=\frac{1}{3}$.

Hence $r=\frac{CN}{\sin x}=27$.

19. Answer: 2010

Consider any integer k where $1 \le k \le 2010$. By division algorithm, there exists unique pair of integers (q, r) such that $2010 = kq + r$ with $0 \le r \le k - 1$. We rewrite this as $2010 = (k - r)q + r(q + 1)$. That is, $k - r$ copies of q and r copies of $q + 1$ add up to 2010. Thus there is one desired expression for each value of k, which is clearly unique. Hence there are 2010 such expressions in all.

20. Answer: 60

Let a be a positive integer such that the sum of n consecutive integers $a, a + 1, \ldots, a + (n - 1)$ is 2010, that is,

$$a+(a+1)+\cdots+(a+n-1)=2010.$$

This gives $\dfrac{n(2a+n-1)}{2}=2010$, or

$$n(2a+n-1)=4020=2^2\times3\times5\times67. \qquad (1)$$

Since $n < 2a + n - 1$, we have

$$n<\sqrt{2^2\times3\times5\times67}=2\sqrt{1005}<2\times32=64. \qquad (2)$$

Now (1) and (2) imply that $n\in\{1,2,3,4,5,6,10,12,15,20,30,60\}$. Since n and $2a + n - 1$ have different parities, it follows that n and $\dfrac{4020}{n}$ have different parities. Consequently, we have

$$n\in\{1,3,4,5,12,15,20,60\}.$$

If $n = 60$, then $2a+n-1=\dfrac{4020}{60}=67$, so $a = 4$. Thus the largest possible value of n is 60.

21. Answer: 2

First note that if $n \ge 4$, then

$$\begin{aligned}1!+2!+3!+\cdots+n! &\equiv 1!+2!+3!+4! \pmod 5\\ &\equiv 1+2+1+4\equiv 3 \pmod 5.\end{aligned}$$

Since the square of any integer is congruent to either 1 or 4 modulo 5, it follows that $1! + 2! + 3! + \cdots + n! \neq m^2$ for any integer m in this case. So we consider $n < 4$.
Now we have

$$1! = 1^2,\ \ 1! + 2! = 3,\ \ 1! + 2! + 3! = 3^2.$$

Hence we conclude that there are two pairs of positive integers (n, m), namely, $(1, 1)$ and $(3, 3)$, such that $1! + 2! + 3! + \cdots + n! = m^2$.

22. Answer: 65

Since BE bisects $\angle CBA$, we have $\dfrac{BC}{BA}=\dfrac{EC}{EA}=\dfrac{119}{169}$. Thus we can let $BC=119y$ and $BA = 169y$ for some real number y. Since $\angle BCA = 90^{\circ}$, we have

$$\begin{aligned}
AB^2 &= AC^2 + BC^2\\
(169y)^2 &= (169+119)^2+(119y)^2\\
y^2(169-119)(169+119) &= (169+119)^2\\
y^2 &= \frac{169+119}{169-119}=\frac{144}{25}\\
y &= \frac{12}{5}
\end{aligned}$$

Hence, from triangle BCE, we have $BE = \sqrt{119^2+(119y)^2} = 119\times\dfrac{13}{5}$.

Finally, note that ΔADE and ΔBCE are similar, so we have

$$ED=\frac{AE\times CE}{BE}=\frac{169\times 119}{119\times\frac{13}{5}}=65\,\text{cm}.$$

23. Answer: 72

First we find the number of ordered pairs (m, n) of positive integers m and n such that $m + n = 190$ and m and n are not relatively prime.

To this end, write $m = ka$ and $n = kb$, where k, a and b are positive integers with $k > 1$. Since $m + n = 190$, we see that k is a factor of $190 = 2\times 5\times 19$ with $k \neq 190$. We consider six cases:

(i) $k = 2$. Then $a + b = 95$, and there are 94 such pairs (a, b) of a and b such that the equation holds.

(ii) $k = 5$. Then $a + b = 38$, and there are 37 such pairs (a, b) of a and b such that the equation holds.

(iii) $k = 19$. Then $a + b = 10$, and there are 9 such pairs (a, b) of a and b such that the equation holds.

(iv) $k = 10$. Then $a + b = 19$, and there are 18 such pairs (a, b) of a and b such that the equation holds.

(v) $k = 38$. Then $a + b = 5$, and there are 4 such pairs (a, b) of a and b such that the equation holds.

(vi) $k = 95$. Then $a + b = 2$, and there is 1 such pair (a, b) of a and b such that the equation holds.

It follows from the above cases that the number of ordered pairs (m, n) of positive integers m and n such that $m + n = 190$ and m and n are not relatively prime is $94 + 37 + 9 - 18 - 4 - 1 = 117$.

Since the total number of ordered pairs (m, n) such that $m + n = 190$ is 189, we conclude that the required number of ordered pairs (m, n) where m and n are relatively prime is 189 – 117 = 72.

24. Answer: 32

For all real values of x for which $f(x)$ is defined, we have

$$\begin{aligned}
f(x) &= \frac{9}{1+\cos 2x} + \frac{25}{1-\cos 2x} = \frac{9}{2\cos^2 x} + \frac{25}{2\sin^2 x} \\
&= \frac{1}{2}\left(9\tan^2 x + 9\right) + \frac{1}{2}\left(25\cot^2 x + 25\right) \\
&= 17 + \frac{1}{2}\left(9\tan^2 x + 25\cot^2 x\right) \\
&\geq 17 + \frac{1}{2}\left(2\sqrt{\left(9\tan^2 x\right)\left(25\cot^2 x\right)}\right) \quad \text{(AM-GM Inequality)} \\
&= 17 + \frac{1}{2}\left(2\sqrt{9\times 25}\right) \\
&= 32.
\end{aligned}$$

Note that $f(\tan^{-1}\sqrt{\frac{5}{3}}) = 32$. Thus the least possible value of $f(x)$ is 32.

25. Answer: 2002

First we place the 5 red balls on a straight line and then place 1 blue ball between 2 adjacent red balls. With this arrangement fixed, the condition of the question is satisfied. We are now left with 9 blue balls. We can place the remaining 9 blue balls into the spaces before, after or in between the 5 red balls. The number of ways that this can be done is the answer to the question. Including the two ends, there are 4 + 2 = 6 spaces into which these 9 blue balls can be placed. The number of ways of distributing the 9 blue balls into the 6 spaces is

$$\binom{9+6-1}{9} = \binom{14}{9} = \binom{14}{5} = 2002.$$

26. Answer: 49

Consider the following subsets of S:

$$\begin{aligned}
S_1 &= \{1, 2, 3, \ldots, 49, 50\}, \\
S_2 &= \{51, 52, 53, \ldots, 99, 100\}, \\
S_3 &= \{101, 102, 103, \ldots, 149, 150\}, \\
&\vdots \\
S_{2000} &= \{99951, 99952, 99953, \ldots, 99999, 100000\}.
\end{aligned}$$

In other words, $S_i = \{50i - 49, 50i - 48, \ldots, 50i\}$ for $i = 1, 2, \ldots, 2000$. Note that S is partitioned into these subsets $S_1, S_2, S_3, \ldots, S_{2000}$.

By Pigeonhole Principle, for any subset A of S with $|A| = 2010$, there exists i, where $1 \leq i \leq 2000$, such that $|A \cap S_i| \geq 2$.
Let $a, b \in A \cap S_i$. It is clear that $|a - b| \leq 49$.
To show that 49 is the least possible value of k, we find a subset $A \subseteq S$ with $|A| = 2010$ such that $|a - b| \geq 49$ for any distinct $a, b \in A$. Let

$$A = \{49j + 1 : j = 0, 1, 2, ..., 2009\} = \{1, 50, 99, 148, ..., 98442\}.$$

Then A is a subset of S with $|A| = 2010$ and $|a - b| \geq 49$ for any distinct $a, b \in A$.

27. Answer: 112

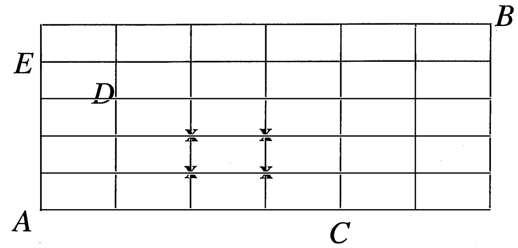


Figure 1

We observe that to avoid the four points marked x, the path must cross either C, D or E as shown in Figure 1 above. Further, the paths that cross C, D or E are exclusive, that is, no path can cross both C and D or D and E, or C and E.
There is only 1 way to get from A to C and from A to E. It is easy to see that there are 4 ways to get from A to D.

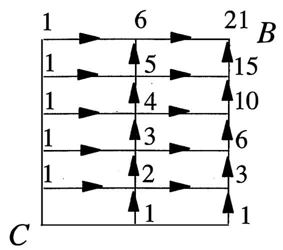


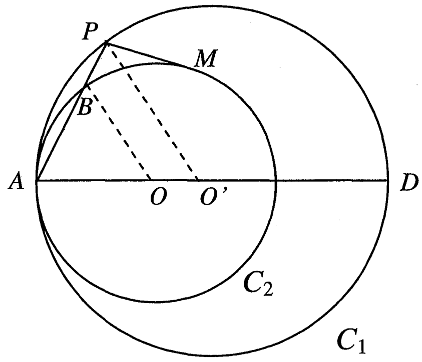
Figure 2

To count the number of ways to get from either C, D or E to B, we note that the number of ways to get to a certain junction is the sum of the numbers of ways to get to the two junctions immediately preceding it from the left or from below (as shown in Figure 2). Therefore there are 21 ways to get from C to B. Similarly, there are 21 ways to get from D to B and 7 ways to get from E to B.
Hence the number of ways to get from A to B that

- pass through C: number of ways from A to C × number of ways from C to $B = 1 \times 21 = 21$;
- pass through D: number of ways from A to D × number of ways from D to $B = 4 \times 21 = 84$;
- pass through E: number of ways from A to E × number of ways from E to $B = 1 \times 7 = 7$.

It follows that the total number of ways from A to B is $21 + 84 + 7 = 112$.

28. Answer: 60



Let O be the centre of C_2 and let PA intersect C_2 at B. The homothety centred at A mapping C_2 to C_1 has similitude ratio $\frac{8}{10}$. It maps B to P. Thus $\frac{AB}{AP} = \frac{8}{10}$. (This can also be seen by connecting P to the centre O' of C_1 so that the triangles ABO and APO' are similar.) The power of P with respect to C_2 is $PM^2 = 20$. Thus $PB \cdot PA = 20$, or equivalently $(PA - AB)PA = 20$. Together with $\frac{AB}{AP} = \frac{8}{10}$, we obtain $AB = 8$ and $AP = 10$. Consequently, the triangle ABO is equilateral, and hence $\angle PAD = \angle BAO = 60^\circ$.

29. Answer: 6
We shall show that $a = 3$, $b = 2$ and $c = 1$.
Note that $2a > b + c$. As $b + c$ is a multiple of a, it follows that $a = b + c$.
Let $a + c = kb$. Then $kb = a + c = b + c + c$, so $2c = (k - 1)b$. Since $c < b$, we must have $k = 2$ and therefore $b = 2c$. Since b and c are relatively prime, this implies that $c = 1$ and $b = 2$. Thus $a = 3$. Hence $abc = 6$.

30. Answer: 680
For any 3-element subset $\{a, b, c\}$, define a mapping f by

$$f(\{a, b, c\}) = \{a, b-1, c-3\}.$$

Now observe that $\{a, b, c\}$ is a subset of $\{1, 2, 3, 4, \ldots, 20\}$ with $a < b - 1 < c - 3$ if and only if $f(\{a, b, c\})$ is a 3-element subset of $\{1, 2, 3, \ldots, 17\}$. Hence the answer is $\binom{17}{3} = 680$.

31. Answer: 2780

Note that $0 \le f(n) \le 4$ for $1 \le n \le 99999$. For $k = 0, 1, 2, 3, 4$, let a_k denote the number of integers n, where $1 \le n \le 99999$, such that $f(n) = k$. Then

$$M = \sum_{k=0}^{4} k a_k 2^k = \sum_{k=1}^{4} k a_k 2^k.$$

By considering the number of 2-digit, 3-digit, 4-digit and 5-digit positive integers with exactly one 0 in their decimal representation, we obtain

$$a_1 = 9 + 9\times 9\times 2 + 9\times 9\times 9\times 3 + 9\times 9\times 9\times 9\times 4 = 28602.$$

Similarly, we have

$$a_2 = 9 + 9\times 9\times \binom{3}{2} + 9\times 9\times 9\times \binom{4}{2} = 4626,$$

$$a_3 = 9 + 9^2 \times 4 = 333,$$

$$a_4 = 9.$$

Hence

$$M = 1\times 28602\times 2 + 2\times 4626\times 2^2 + 3\times 333\times 2^3 + 4\times 9\times 2^4 = 102780,$$

and it follows that $M - 100000 = 2780$.

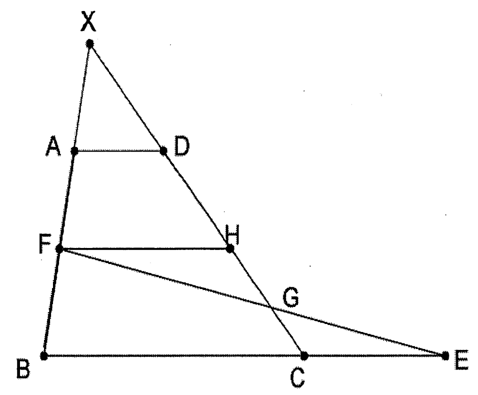
32. Answer: 5

Let $n = p^4 - 5p^2 + 13$. When $p = 3$, we have $n = 49$ and so the sum of digits is 13. When $p = 5$, we have $n = 513$ and the resulting sum of digits is 9.
Now let $p > 5$ be a prime. We have

$$n = p^4 - 5p^2 + 13 = (p-2)(p-1)(p+1)(p+2) + 9.$$

Since $p - 2, p - 1, p, p + 1$ and $p + 2$ are five consecutive integers, at least one of them is divisible by 5. Since $p \neq 5$, we must have 5 divides one of the numbers $p - 2, p - 1, p + 1$ and $p + 2$, so 5 divides the product $(p - 2)(p - 1)(p + 1)(p + 2)$. Observe that at least one of the numbers $p + 1$ and $p + 2$ is even. Therefore we see that $(p - 2)(p - 1)(p + 1)(p + 2)$ is divisible by 10. It follows that for any prime $p > 5$, the number n has at least two digits and the units digit is 9, so the sum of the digits is greater than 9. Consequently the smallest possible sum of digits is 9, and this value is attained when $p = 5$.

33. Answer: 360



Extend BA and CD to meet at X. Let H be the point on CD such that $FH // BC$.

Let $AD = CE = a$. Then $BC = 3a$, and $FH = \frac{1}{2}(AD + BC) = 2a$.

By the similarity of triangles FHG and ECG, we have

(i) area of $\Delta FHG = \left(\frac{FH}{CE}\right)^2 \times$ area of $\Delta ECG = 60$ cm^2;

(ii) $\frac{HG}{CG} = \frac{FH}{CE} = 2$, so that $HG = 2CG$ and $DH = HC = \frac{3}{2}HG$.

It follows from (i) and (ii) that the area of triangle $FDH = \frac{3}{2} \times 60 = 90$ cm^2.

Now, let area of triangle XAD be y cm^2. By the similarity of triangles XAD and XFH, we have

(iii) $\frac{XA}{XF} = \frac{AD}{FH} = \frac{1}{2}$, so that $XA = AF$ and hence
area of $\Delta XDF = 2\times$ area of $\Delta XAD = 2y$ cm^2;

(iv) area of $\Delta XFH = \left(\frac{FH}{AD}\right)^2 y = 4y$.

It follows from (iii) and (iv) that the area of triangle $FDH = 4y - 2y = 2y$ cm^2.

Since the area of triangle FDH is 90 cm^2, we get $y = 45$.

Finally, by the similarity of triangles XAD and ABC,

$$\text{area of } \Delta XBC = \left(\frac{BC}{AD}\right)^2 y = 9y.$$

Hence the area of trapezium $ABCD = 8y = 360$ cm^2.

34. Answer: 75

First we note that every positive integer m can be written uniquely (in base 27) as

$$m = b_0 + b_1 \times 27 + b_2 \times 27^2 + \cdots + b_r \times 27^r,$$

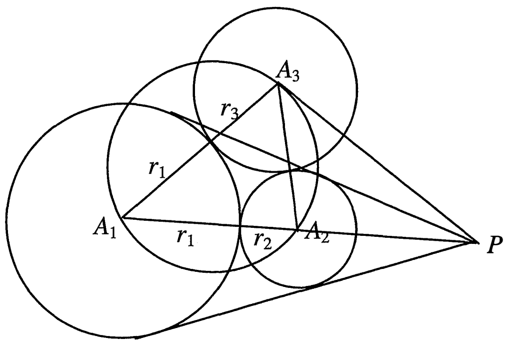
where r and $b_0, b_1, b_2, \ldots, b_r$ (depending on m) are non-negative integers with $b_i < 27$ for $i = 0, 1, 2, \ldots, r$.

Since the coefficients of $P(x)$ are non-negative integers and $P(1) = 25$, we see that $a_k \le 25 < 27$ for $0 \le k \le n$. Thus by the above remark, the polynomial $P(x)$ is uniquely determined by the value $P(27) = 1771769$. Writing 1771769 in base 27, we obtain $1771769 = 2 + 11 \times 27 + 9 \times 27^3 + 3 \times 27^4$. Therefore

$$P(x) = 2 + 11x + 9x^3 + 3x^4.$$

Hence $a_0 + 2a_1 + 3a_2 + \cdots + (n+1)a_n = 2 + 2 \times 11 + 4 \times 9 + 5 \times 3 = 75.$

35. Answer: 12



First we shall show that $r_3 = \sqrt{r_1 r_2}$. Let P be the point of concurrence of the tangent to the circumcircle of the triangle $A_1A_2A_3$ at A_3 and the two external common tangents of Γ_1 and Γ_2. Note that the line joining A_1 and A_2 also passes through P.

First we have $PA_3^2 = PA_2 \cdot PA_1$, so $\left(\frac{PA_3}{PA_2}\right)^2 = \frac{PA_1}{PA_2} = \frac{r_1}{r_2}$. That is, $\frac{PA_3}{PA_2} = \sqrt{\frac{r_1}{r_2}}$.

On the other hand,

$$\frac{PA_3}{PA_2} = \frac{\sin \angle A_3A_2P}{\sin \angle A_2A_3P} = \frac{\sin A_2}{\sin A_1} = \frac{r_1 + r_3}{r_2 + r_3}.$$

Thus $\sqrt{\frac{r_1}{r_2}} = \frac{r_1 + r_3}{r_2 + r_3}$. Solving for r_3, we obtain $r_3 = \sqrt{r_1 r_2}$. Substituting $r_1 = 18$, $r_2 = 8$, we get $r_3 = 12$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Senior Section, Round 2)

Saturday, 25 June 2010 **0930-1230**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

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1. In the triangle ABC with $AC > AB$, D is the foot of the perpendicular from A onto BC and E is the foot of the perpendicular from D onto AC. Let F be the point on the line DE such that $EF \cdot DC = BD \cdot DE$. Prove that AF is perpendicular to BF.

2. The numbers $\frac{1}{1}, \frac{1}{2}, \cdots, \frac{1}{2010}$ are written on a blackboard. A student chooses any two of the numbers, say x, y, erases them and then writes down $x + y + xy$. He continues to do this until only one number is left on the blackboard. What is this number?

3. Given $a_1 \geq 1$ and $a_{k+1} \geq a_k + 1$ for all $k = 1, 2, \cdots, n$, show that

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq (a_1 + a_2 + \cdots + a_n)^2.$$

4. An infinite sequence of integers, $a_0, a_1, a_3 \ldots$, with $a_0 > 0$, has the property that for any $n \geq 0$, $a_{n+1} = a_n - b_n$, where b_n is the number having the same sign as a_n, but having the digits written in the reverse order. For example if $a_0 = 1210$, $a_1 = 1089$ and $a_2 = -8712$, etc. Find the smallest value of a_0 so that $a_n \neq 0$ for all $n \geq 1$.

5. Let p be a prime number and let $a_1, a_2, \ldots, a_k$ be distinct integers chosen from $1, 2, \ldots, p-1$. For $1 \leq i \leq k$, let $r_i^{(n)}$ denote the remainder of the integer na_i upon division by p, so $0 \leq r_i^{(n)} < p$. Define

$$S = \{n : 1 \leq n \leq p-1,\ r_1^{(n)} < \cdots < r_k^{(n)}\}.$$

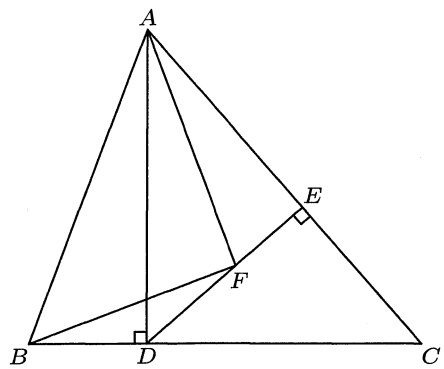
Show that S has less than $\frac{2p}{k+1}$ elements.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Senior Section, Round 2 solutions)

1. Since we are supposed to prove $\angle AFB = 90^\circ$, it means that the 4 points A, B, D, F are concyclic. Note that $AC > AB$ implies that $\angle B > \angle C$. If TD is the tangent to the circumcircle ω of the triangle ABD with B and T lying opposite sides of the line AD, then $\angle ADT = \angle B > \angle C = \angle ADE$ so that ω intersects the interior of DE at F. Therefore F can only be in the interior of DE. Now observe that the triangles ADE and DCE are similar so that $AD/AE = DC/DE$. By the given condition, this can be written as $AD/AE = BD/EF$. This means the triangles ABD and AFE are similar. Thus $\angle ABD = \angle AFE$. This shows that A, B, D, F are concyclic. Therefore $\angle AFB = \angle ADB = 90^\circ$.



2. See Junior Section Question 5.

3. We will prove it by induction. First, it is clear that $a_1^3 \geq a_1^2$ since $a_1 \geq 1$. Next, suppose it is true for n terms. Then

$$\begin{aligned}\sum_{k=1}^{n+1} a_k^3 &\geq a_{n+1}^3 + \sum_{k=1}^{n} a_k^3 \geq a_{n+1}^3 + \left(\sum_{k=1}^{n} a_k\right)^2 \\ &= \left(\sum_{k=1}^{n+1} a_k\right)^2 + a_{n+1}^3 - a_{n+1}^2 - 2a_{n+1}\sum_{k=1}^{n} a_k.\end{aligned}$$

To complete the induction, we'll now show that $a_{n+1}^3 - a_{n+1}^2 - 2a_{n+1}\sum_{k=1}^n a_k \geq 0$. Since $a_{k+1} - a_k \geq 1$, we have $a_{k+1}^2 - a_k^2 \geq a_{k+1} + a_k$. Summing up over $k = 1, \ldots, n$, and using $a_1^2 - a_1 \geq 0$, we have

$$a_{n+1}^2 - a_1^2 \geq a_{n+1} + 2\sum_{k=1}^n a_k - a_1 \quad \Rightarrow \quad a_{n+1}^3 - a_{n+1}^2 - 2a_{n+1}\sum_{k=1}^n a_k \geq 0.$$

4. If a_0 has a single digit, then $a_1 = 0$. Thus a_0 has at least 2 digits. If $a_0 = \overline{ab} = 10a+b$, then $a_1 = 9(a-b)$ which is divisible by 9. it follows that all subsequent terms are divisible by 9. Checking all 2-digit multiples of 9 shows that eventually 9 appears (Note that $\overline{ab}$ and $\overline{ba}$ give rise to the same sequence, but with opposite signs):

$$81 \to 63 \to 27 \to 45 \to 9.$$

If $a_0 = \overline{abc}$, then $a_1 = 99(a-c)$. Thus if suffices to investigate 3-digit multiples of 99, i.e., $198, \ldots, 990$. Here we find that 99 will eventually appear:

$$990 \to 891 \to 693 \to 297 \to -495 \to 99.$$

If $a_0 = \overline{abcd}$, then $a_1 = 999(a-d)+90(b-c)$. If b, c are both 0, then a_1 and all subsequent terms are multiples of 999. However, if such numbers appear in the sequence, eventually 999 will appear:

$$9990 \to 8991 \to 6993 \to 2997 \to -4995 \to 999.$$

For 1010, we get 909 and for 1011 we get -90. For 1012, we get

$$1012 \to -1089 \to -8712 \to 6534 \to 2178 \to -6534$$

and the sequence becomes periodic thereafter. Thus the smallest $a_0 = 1012$.

5. Let $r_0^{(n)} = 0$ and $r_{k+1}^{(n)} = p$. Set

$$S' = \{n : 1 \leq n \leq p-1, \sum_{i=0}^{k} |r_{i+1}^{(n)} - r_i^{(n)}| = p\}.$$

Note that

$$\sum_{i=0}^{k} |r_{i+1}^{(n)} - r_i^{(n)}| = p \quad \text{iff} \quad r_0^{(n)} \leq r_1^{(n)} \leq \cdots \leq r_{k+1}^{(n)}.$$

Thus $|S| = |S'|$. Since for $n \in S'$, $|r_{i+1}^{(n)} - r_i^{(n)}| = r_{i+1}^{(n)} - r_i^{(n)} \equiv n(a_{i+1} - a_i) \pmod{p}$ and $p \nmid (a_{i+1} - a_i)$, the numbers $r_{i+1}^{(n)} - r_i^{(n)}$, $1 \leq n \leq p-1$, are all distinct. Therefore

$$p|S'| = \sum_{n \in S'} \sum_{i=0}^{k} |r_{i+1}^{(n)} - r_i^{(n)}| \geq \sum_{i=0}^{k} \sum_{n \in S'} |r_{i+1}^{(n)} - r_i^{(n)}| \geq \sum_{i=0}^{k} \sum_{j=1}^{|S'|} j \geq \frac{(k+1)|S'|(|S'|+1)}{2}.$$

Therefore $|S| < \frac{2p}{k+1}$.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2010
(Open Section, Round 1)

Wednesday, 2 June 2010 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

1. Let S be the set of all integers n such that $\dfrac{8n^3 - 96n^2 + 360n - 400}{2n - 7}$ is an integer. Find the value of $\sum_{n \in S} |n|$.

2. Determine the largest value of x for which

$$\left|x^2 - 4x - 39601\right| \geq \left|x^2 + 4x - 39601\right|.$$

3. Given that

$$x = \lfloor 1^{1/3} \rfloor + \lfloor 2^{1/3} \rfloor + \lfloor 3^{1/3} \rfloor + \cdots + \lfloor 7999^{1/3} \rfloor,$$

find the value of $\lfloor \frac{x}{100} \rfloor$, where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. (For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 30 \rfloor = 30$, $\lfloor -10.5 \rfloor = -11$.)

4. Determine the smallest positive integer C such that $\dfrac{6^n}{n!} \leq C$ for all positive integers n.

5. Let CD be a chord of a circle Γ_1 and AB a diameter of Γ_1 perpendicular to CD at N with $AN > NB$. A circle Γ_2 centred at C with radius CN intersects Γ_1 at points P and Q, and the segments PQ and CD intersect at M. Given that the radii of Γ_1 and Γ_2 are 61 and 60 respectively, find the length of AM.

6. Determine the minimum value of $\sum_{k=1}^{50} x_k$, where the summation is done over all possible positive numbers $x_1, \ldots, x_{50}$ satisfying $\sum_{k=1}^{50} \dfrac{1}{x_k} = 1$.

7. Find the sum of all positive integers p such that the expression $(x - p)(x - 13) + 4$ can be expressed in the form $(x + q)(x + r)$ for distinct integers q and r.

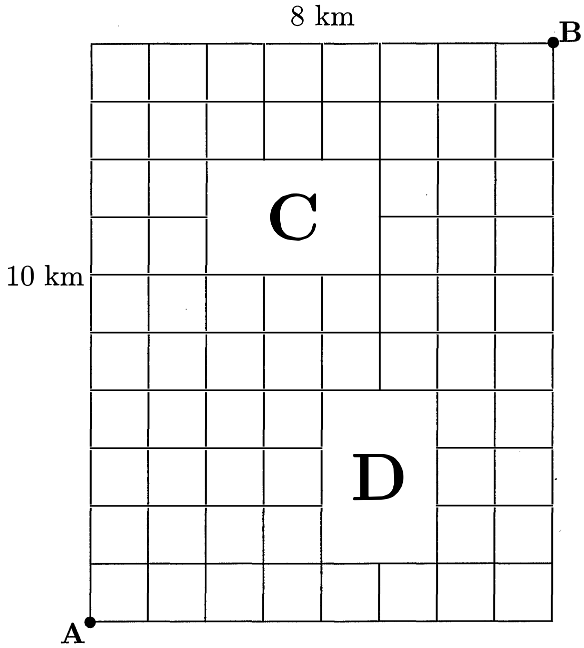
8. Let $p_k = 1 + \dfrac{1}{k} - \dfrac{1}{k^2} - \dfrac{1}{k^3}$, where k is a positive integer. Find the least positive integer n such that the product $p_2 p_3 \cdots p_n$ exceeds 2010.

9. Let B be a point on the circle centred at O with diameter AC and let D and E be the circumcentres of the triangles OAB and OBC respectively. Given that $\sin \angle BOC = \frac{4}{5}$ and $AC = 24$, find the area of the triangle BDE.

10. Let f be a real-valued function with the rule $\mathrm{f}(x) = x^3 + 3x^2 + 6x + 14$ defined for all real value of x. It is given that a and b are two real numbers such that $\mathrm{f}(a) = 1$ and $\mathrm{f}(b) = 19$. Find the value of $(a + b)^2$.

11. If $\cot \alpha + \cot \beta + \cot \gamma = -\dfrac{4}{5}$, $\tan \alpha + \tan \beta + \tan \gamma = \dfrac{17}{6}$ and $\cot \alpha \cot \beta + \cot \beta \cot \gamma + \cot \gamma \cot \alpha = -\dfrac{17}{5}$, find the value of $\tan(\alpha + \beta + \gamma)$.

12. The figure below shows a road map connecting two shopping malls A and B in a certain city. Each side of the smallest square in the figure represents a road of distance 1km. Regions C and D represent two large residential estates in the town. Find the number of shortest routes to travel from A to B along the roads shown in the figure.



13. Let $a_1 = 1$, $a_2 = 2$ and for all $n \geq 2$, $a_{n+1} = \dfrac{2n}{n+1}a_n - \dfrac{n-1}{n+1}a_{n-1}$. It is known that $a_n > 2 + \dfrac{2009}{2010}$ for all $n \geq m$, where m is a positive integer. Find the least value of m.

14. It is known that

$$\sqrt{9 - 8\sin 50^\circ} = a + b\sin c^\circ$$

for exactly one set of positive integers (a, b, c), where $0 < c < 90$. Find the value of $\dfrac{b+c}{a}$.

15. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, find the value of $\lfloor \alpha^6 \rfloor$, where $\lfloor x \rfloor$ is the least positive integer not exceeding x.

16. If a positive integer cannot be written as the difference of two square numbers, then the integer is called a "cute" integer. For example, 1, 2 and 4 are the first three "cute" integers. Find the 2010^{th} "cute" integer.

(Note: A *square number* is the square of a positive integer. As an illustration, 1, 4, 9 and 16 are the first four square numbers.)

17. Let $f(x)$ be a polynomial in x of degree 5. When $f(x)$ is divided by $x-1$, $x-2$, $x-3$, $x-4$ and $x^2 - x - 1$, $f(x)$ leaves a remainder of $3, 1, 7, 36$ and $x - 1$ respectively. Find the square of the remainder when $f(x)$ is divided by $x + 1$.

18. Determine the number of ordered pairs of positive integers (a, b) satisfying the equation

$$100(a+b) = ab - 100.$$

(Note: As an illustration, $(1, 2)$ and $(2, 1)$ are considered as two distinct ordered pairs.)

19. Let $p = a^b + b^a$. If a, b and p are all prime, what is the value of p?

20. Determine the value of the following expression:

$$\left\lfloor \frac{11}{2010} \right\rfloor + \left\lfloor \frac{11\times 2}{2010} \right\rfloor + \left\lfloor \frac{11\times 3}{2010} \right\rfloor + \left\lfloor \frac{11\times 4}{2010} \right\rfloor + \cdots + \left\lfloor \frac{11\times 2009}{2010} \right\rfloor,$$

where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

21. Numbers $1, 2, \ldots, 2010$ are placed on the circumference of a circle in some order. The numbers i and j, where $i \neq j$ and $i, j \in \{1, 2, \ldots, 2010\}$ form a *friendly* pair if

(i) i and j are not neighbours to each other, and

(ii) on one or both of the arcs connecting i and j along the circle, all numbers in between them are greater than both i and j.

Determine the minimal number of *friendly* pairs.

22. Let S be the set of all non-zero real-valued functions f defined on the set of all real numbers such that

$$\mathrm{f}(x^2 + yf(z)) = x\mathrm{f}(x) + z\mathrm{f}(y)$$

for all real numbers x, y and z. Find the maximum value of f(12345), where $\mathrm{f} \in S$.

23. All possible 6-digit numbers, in each of which the digits occur in non-increasing order from left to right (e.g., 966541), are written as a sequence in increasing order (the first three 6-digit numbers in this sequence are 100000, 110000, 111000 and so on). If the 2010^{th} number in this sequence is denoted by p, find the value of $\lfloor \frac{p}{10} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

24. Find the number of permutations $a_1a_2a_3a_4a_5a_6$ of the six integers from 1 to 6 such that for all i from 1 to 5, a_{i+1} does not exceed a_i by 1.

25. Let

$$A = \left(\binom{2010}{0} - \binom{2010}{-1}\right)^2 + \left(\binom{2010}{1} - \binom{2010}{0}\right)^2 + \left(\binom{2010}{2} - \binom{2010}{1}\right)^2 + \cdots + \left(\binom{2010}{1005} - \binom{2010}{1004}\right)^2.$$

Determine the minimum integer s such that

$$sA \geq \binom{4020}{2010}.$$

(Note: For a given positive integer n, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for $r = 0, 1, 2, 3, \cdots, n$; and for all other values of r, define $\binom{n}{r} = 0$.)

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1. Answer: 50

   Note that $\dfrac{8n^3 - 96n^2 + 360n - 400}{2n-7} = 4n^2 - 34n + 61 + \dfrac{27}{2n-7}$. Since $4n^2 - 34n + 61$ is an integer for all integers n, we must have that 27 divisible by $2n-7$. Hence, $2n - 7 = -1, 1, -3, 3, -9, 9, -27, 27$, so that $n = 3, 4, 2, 5, -1, 8, -10, 17$. Hence the required sum equals 50.

2. Answer: 199

   By direct computation,

$$\begin{aligned}
&\left|x^2 - 4x - 39601\right| \geq \left|x^2 + 4x - 39601\right| \\
\iff & (x^2 - 4x - 39601)^2 - (x^2 + 4x - 39601)^2 \geq 0 \\
\iff & 2(x^2 - 39601)(-8x) \geq 0 \\
\iff & x(x+199)(x-199) \leq 0,
\end{aligned}$$

   we conclude that the largest possible value of x is 199.

3. Answer: 1159

   Note that

$$\begin{aligned}
x &= \lfloor 1^{1/3} \rfloor + \lfloor 2^{1/3} \rfloor + \lfloor 3^{1/3} \rfloor + \cdots + \lfloor 7999^{1/3} \rfloor \\
&= \sum_{1^3 \leq k < 2^3} \lfloor k^{1/3} \rfloor + \sum_{2^3 \leq k < 3^3} \lfloor k^{1/3} \rfloor + \sum_{3^3 \leq k < 4^3} \lfloor k^{1/3} \rfloor + \cdots + \sum_{19^3 \leq k < 20^3} \lfloor k^{1/3} \rfloor \\
&= \sum_{1^3 \leq k < 2^3} 1 + \sum_{2^3 \leq k < 3^3} 2 + \sum_{3^3 \leq k < 4^3} 3 + \cdots + \sum_{19^3 \leq k < 20^3} 19 \\
&= \left(2^3 - 1^3\right) + 2\left(3^3 - 2^3\right) + 3\left(4^3 - 3^3\right) + \cdots + 18\left(19^3 - 18^3\right) + 19\left(20^3 - 19^3\right) \\
&= 19(8000) - \sum_{k=1}^{19} k^3 \\
&= 19(8000) - \left(\frac{19 \times 20}{2}\right)^2 \\
&= 115900
\end{aligned}$$

   $\therefore \lfloor \frac{x}{100} \rfloor = 1159.$

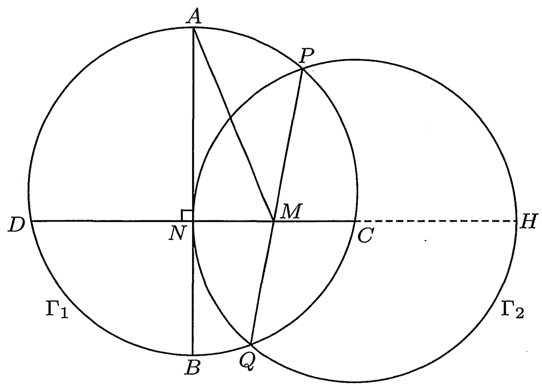
4. Answer: 65

Define $f(n) = \dfrac{6^n}{n!}$ for $n = 1, 2, 3, \cdots$. It is clear that $f(1) = 6, f(2) = 18, f(3) = 36, f(4) = 54$ and $f(5) = 64.8$. For all $n \geq 6$,

$$f(n) = \frac{6^n}{n!} \leq \frac{6}{1} \times \frac{6}{2} \times \frac{6}{3} \times \frac{6}{4} \times \frac{6}{5} \times \frac{6}{6} = 6 \times 3 \times 2 \times 1.8 = 64.8$$

5. Answer: 78

Extend DC meeting Γ_2 at H. Note that $DN = NC = CH = 60$. Since M is of equal power with respect to Γ_1 and Γ_2. Thus $MN \cdot MH = MC \cdot MD$. That is $MN(MC + 60) = MC(MN + 60)$ giving $MN = MC$. Thus $MN = 30$.



The power of N with respect to Γ_1 is $DN \cdot NC = 60^2$, and is also equal to $NA \cdot NB = NA \cdot (AB - NA) = NA \cdot (122 - NA)$. Thus $NA \cdot (122 - NA) = 60^2$. Solving this quadratic equation, we get $NA = 72$ or 50. Since $NA > NB$, we have $NA = 72$. Consequently $AM = \sqrt{NA^2 + MN^2} = \sqrt{72^2 + 30^2} = 78$.

6. Answer: 2500

By Cauchy-Schwarz inequality,

$$\sum_{k=1}^{50} 1 \leq \sqrt{\sum_{k=1}^{50} x_k \sum_{k=1}^{50} \frac{1}{x_k}},$$

and equality holds if and only if $x_k = 50$ for $k = 1, \ldots, 50$. Therefore the required value is 2500.

7. Answer: 26

Let $(x-p)(x-13)+4 = (x+q)(x+r)$. Substituting $x = -q$ into the above identity yields $(-q-p)(-q-13) = -4$, which becomes $(q+p)(q+13) = -4$. Since p and q are integers, we must have the following cases:

(a) $q + p = 4, q + 13 = -1$;

(b) $q + p = -4, q + 13 = 1$;

(c) $q + p = 2, q + 13 = -2$; or

(d) $q + p = -2, q + 13 = 2$

For case (a), we obtain $q=-14, p=8$ and hence $(x-p)(x-13)+4=(x-14)(x-17)$;

For case (b), we obtain $q=-12, p=8$ and hence $(x-p)(x-13)+4=(x-9)(x-12)$;

For case (c), we obtain $q=-15, p=17$ and hence $(x-p)(x-13)+4=(x-15)^2$; which is NOT what we want;

For case (d), we obtain $q=-11, p=9$ and hence $(x-p)(x-13)+4=(x-11)^2$; which is also NOT what we want. Hence the two possible values of p are 8 and 18, the sum of which is 26.

8. Answer: 8038

First, note that

$$p_k = 1+\frac{1}{k}-\frac{1}{k^2}-\frac{1}{k^3}=\left(1-\frac{1}{k}\right)\left(1+\frac{1}{k}\right)^2=\frac{(k-1)(k+1)^2}{k^3}.$$

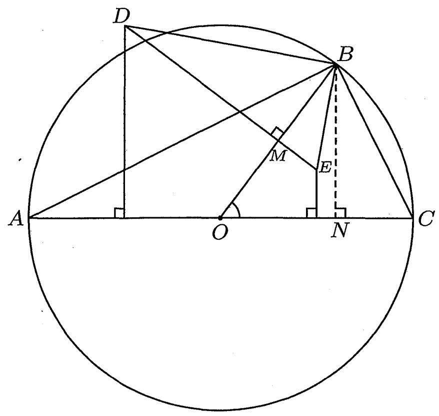
Therefore,

$$p_2p_3\cdots p_n=\frac{1\cdot 3^2}{2^3}\cdot\frac{2\cdot 4^2}{3^3}\cdot\frac{3\cdot 5^2}{4^3}\cdots\frac{(n-1)(n+1)^2}{n^3}=\frac{(n+1)^2}{4n}.$$

Next, observe that $\dfrac{(n+1)^2}{4n}>2010$ is equivalent to $n^2-8038n+1>0$, which is equivalent to $n(n-8038)>-1$. Since n is a positive integer, the last inequality is satisfied if and only if $n\geq 8038$. Consequently, the least n required is 8038.

9. Answer: 45

Let $d=AC=24$. First, it is not difficult to see that $\angle DEB=\angle ACB$ and $\angle EDB=\angle CAB$, so that the triangles DBE and ABC are similar.



Let M and N be the feet of the perpendiculars from B onto DE and AC respectively. As M is the midpoint of OB, we have $BM=\frac{d}{4}$. Also $BN=BO\sin\angle BOC=\frac{d}{2}\times\frac{4}{5}=\frac{2d}{5}$. Therefore $DE=AC\times\frac{BM}{BN}=d\times\frac{5d}{8d}=\frac{5d}{8}$. Thus the area of the triangle BDE is $\frac{1}{2}\times BM\times DE=\frac{1}{2}\times\frac{d}{4}\times\frac{5d}{8}=\frac{5d^2}{64}$. Substituting $d=24$, the area of the triangle BDE is 45.

10. Answer: 4

We note that $f(x) = (x+1)^3 + 3(x+1) + 10$. Let $g(y) = y^3 + 3y$, which is a strictly increasing odd function. Hence $f(a) = 1$ implies $g(a+1) = -9$ and $f(b) = 19$ implies $g(b+1) = 9$. Thus, $a+1 = -(b+1)$, implying $a+b = -2$, so that $(a+b)^2 = 4$.

11. Answer: 11

Let $x = \tan\alpha$, $y = \tan\beta$ and $z = \tan\gamma$. Then

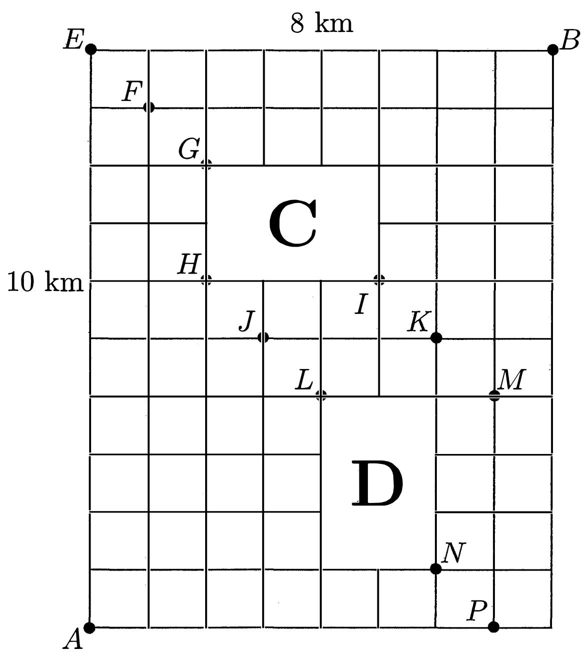
$$\begin{aligned} \frac{xy+yz+zx}{xyz} &= -\frac{4}{5} \\ x+y+z &= \frac{17}{6} \\ \frac{x+y+z}{xyz} &= -\frac{17}{5} \end{aligned}$$

From the above three equations, $xyz = -\frac{5}{6}$ and $xy+yz+zx = \frac{2}{3}$. It follows that

$$\tan(\alpha+\beta+\gamma) = \frac{x+y+z-xyz}{1-(xy+yz+zx)} = \frac{17/6-(-5/6)}{1-2/3} = 11.$$

12. Answer: 22023

Include the points $E, F, G, H, I, J, K, L, M, N$ and P in the diagram as shown and consider all possible routes:



For the route $A \to E \to B$, there is 1 way.

For the route $A \to F \to B$, there are $\binom{10}{1} \cdot \binom{8}{1} = 80$ ways.

For the route $A \to G \to B$, there are $\binom{10}{2} \cdot \binom{8}{2} = 1260$ ways.

For the route $A \to H \to I \to B$, there are $\binom{8}{2} \cdot \binom{7}{3} = 980$ ways.

For the route $A \to J \to I \to B$, there are $\binom{8}{3} \cdot \binom{3}{1} \cdot \binom{7}{3} = 5880$ ways.

For the route $A \to J \to K \to B$, there are $\binom{8}{3} \cdot \binom{7}{2} = 1176$ ways.

For the route $A \to L \to I \to B$, there are $\binom{8}{4} \cdot \binom{3}{1} \cdot \binom{7}{3} = 7350$ ways.

For the route $A \to L \to K \to B$, there are $\binom{8}{4} \cdot \binom{3}{1} \cdot \binom{7}{2} = 4410$ ways.

For the route $A \to L \to M \to B$, there are $\binom{8}{4} \cdot \binom{7}{1} = 490$ ways.

For the route $A \to N \to B$, there are $\binom{7}{1} \cdot \binom{11}{2} = 385$ ways.

For the route $A \to P \to B$, there are $\binom{11}{1} = 11$ ways.

Hence, by adding up, there are altogether 22023 ways.

13. Answer: 4021

Rearranging the recurrence relation yields $a_{n+1} - a_n = \dfrac{n-1}{n+1}(a_n - a_{n-1})$ for $n \geq 2$. Thus, for $n \geq 3$, we have

$$\begin{aligned} a_n - a_{n-1} &= \frac{n-2}{n}(a_{n-1} - a_{n-2}) = \frac{n-2}{n} \cdot \frac{n-3}{n-1}(a_{n-2} - a_{n-3}) \\ &= \cdots = \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdots \frac{2}{4} \cdot \frac{1}{3}(a_2 - a_1) \\ &= \frac{2}{n(n-1)} = \frac{2}{n-1} - \frac{2}{n} \end{aligned}$$

for $n \geq 3$. Using the method of difference, we obtain $a_n = 3 - \dfrac{2}{n}$ for $n \geq 3$. Given that $a_n > 2 + \dfrac{2009}{2010}$, we have

$$3 - \frac{2}{n} > 2 + \frac{2009}{2010} = 3 - \frac{1}{2010},$$

yielding $n > 4020$. Hence the least value of m is 4021.

14. Answer: 14

We have

$$\begin{aligned} 9 - 8\sin 50^\circ &= 8 + 8\sin 10^\circ - 8\sin 10^\circ - 8\sin 50^\circ \\ &= 9 + 8\sin 10^\circ - 8\,(2\sin 30^\circ \cos 20^\circ) \\ &= 9 + 8\sin 10^\circ - 8(1 - 2\sin^2 10^\circ) \\ &= 16\sin^2 10^\circ + 8\sin 10^\circ + 1 \\ &= (1 + 4\sin 10^\circ)^2 \\ \therefore \sqrt{9 - 8\sin 50^\circ} &= 1 + 4\sin 10^\circ. \end{aligned}$$

Thus, $a = 1$, $b = 4$ and $c = 10$, and hence $\dfrac{b+c}{a} = 14$.

15. Answer: 3

It can be easily seen that the given equation has exactly one real root α, since (1) all polynomial equations of degree 5 has at least one real root, and (2) the function $f(x) = x^5 - x^3 + x - 2$ is strictly increasing since $f'(x) = 5x^4 - 3x^2 + 1 > 0$ for all real values of x. It can also be checked that $f(\frac{1}{2}) < 0$ and $f(2) > 0$, so that $\dfrac{1}{2} < \alpha < 2$. This is equivalent to $\alpha^6 < 4$ since

$$\begin{aligned}
\alpha^6 < 4 \qquad & \Longleftrightarrow \alpha^4 - \alpha^2 + 2\alpha < 4 \\
& \Longleftrightarrow \alpha^5 - \alpha^3 + 2\alpha^2 < 4\alpha \\
& \Longleftrightarrow 2\alpha^2 - 5\alpha + 2 < 0 \\
& \Longleftrightarrow \frac{1}{2} < \alpha < 2.
\end{aligned}$$

In addition, we claim that $\alpha^6 \geq 3$ since

$$\begin{aligned}
\alpha^6 \geq 3 \quad & \Longleftrightarrow \quad \alpha^4 - \alpha^2 + 2\alpha \geq 3 \\
& \Longleftrightarrow \quad \alpha^5 - \alpha^3 + 2\alpha^2 - 3\alpha \geq 0 \\
& \Longleftrightarrow \quad 2\alpha^2 - 4\alpha + 2 \geq 0,
\end{aligned}$$

the last inequality is always true. Hence $3 \leq \alpha^6 < 4$, thereby showing that $\lfloor \alpha^6 \rfloor = 3$.

16. Answer: 8030

Any odd number greater than 1 can be written as $2k+1$, where $2k+1 = (k+1)^2 - k^2$. Hence all odd integers greater than 1 are not "cute" integers. Also, since $4m = (m+1)^2 - (m-1)^2$, so that all integers of the form $4m$, where $m > 1$, are not "cute". We claim that all integers of the form $4m + 2$ are "cute". Suppose $4m + 2$ (for $m \geq 1$) is not "cute", then

$$4m + 2 = x^2 - y^2 = (x - y)(x + y)$$

for some integers positive integers x and y. However, $x + y$ and $x - y$ have the same parity, so that $x - y$ and $x + y$ must all be even since $4m + 2$ is even. Hence $4m + 2$ is divisible by 4, which is absurd. Thus, $4m + 2$ is "cute". The first few "cute" integers are $1, 2, 4, 6, 10 \cdots$. For $n > 3$, the n^{th} "cute" integer is $4n - 10$. Thus, the 2010^{th} "cute" integer is $4(2010) - 10 = 8030$.

17. Answer: 841

We have $f(1) = 3$, $f(2) = 1$, $f(3) = 7$, $f(4) = 36$ and

$$f(x) = (x^2 - x - 1)g(x) + (x - 1),$$

where $g(x)$ is a polynomial in x of degree 3. Hence $g(1) = -3$, $g(2) = 0$, $g(3) = 1$, and $g(4) = 3$. Thus

$$\begin{aligned}
f(x) \quad = \quad & (x^2 - x - 1)\left[(-3) \cdot \frac{(x-2)(x-3)(x-4)}{(-1)(-2)(-3)} + (1) \cdot \frac{(x-1)(x-2)(x-4)}{(2)(1)(-1)} \right. \\
& \left. + (3) \cdot \frac{(x-1)(x-2)(x-3)}{(3)(2)(1)}\right] + (x - 1).
\end{aligned}$$

Thus, $f(-1) = -29$, so that its square is 841.

18. Answer: 18

Solving for b, we get $b = \frac{100a+100}{a-100}$. Since a and b are positive, we must have $a > 100$. Let $a = 100 + m$, where m is a positive integer. Thus $b = \frac{100(100+m)+100}{m} = 100 + \frac{10100}{m}$. Therefore m must be a factor of $10100 = 101 \times 2^2 \times 5^2$. Conversely, each factor r of m determines a unique solution $(a, b) = (100 + r, 100 + \frac{10100}{r})$ of the equation $100(a + b) = ab - 100$. There are $18 = (1 + 1)(2 + 1)(2 + 1)$ factors of 10100. Consequently there are 18 solutions of the given equation. In fact, these 18 solutions can be found to be $(a, b) = (101, 10200)$, $(102, 5150)$, $(104, 2625)$, $(105, 2120)$, $(110, 1110)$, $(120, 605)$, $(125, 504)$, $(150, 302)$, $(200, 201)$, $(201, 200)$, $(302, 150)$, $(504, 125)$, $(605, 120)$, $(1110, 110)$, $(2120, 105)$,
$(2625, 104)$, $(5150, 102)$, $(10200, 101)$.

19. Answer: 17

If both a, b are odd prime numbers, then p is even and $p \geq 4$, contradicting the condition that p is prime. Thus $a = 2$ or $b = 2$.

Assume that $a = 2$. Then $b \neq 2$; otherwise, $p = 8$ which is not prime.

Thus b is an odd prime number. Let $b = 2k+1$, where k is an integer greater than 1. Thus

$$p = 2^{2k+1} + (2k + 1)^2 = 2 \times 4^k + (2k + 1)^2.$$

We shall show that $k = 1$.

Suppose that $k \geq 2$. If $k \equiv 1(\text{mod } 3)$, then $b > 3$ and

$$b = 2k + 1 \equiv 0(\text{mod } 3),$$

contradicting the condition that b is prime. If $k \not\equiv 1(\text{mod } 3)$, then

$$p = 2 \times 4^k + (2k + 1)^2 \equiv 2 + 4k^2 + 4k + 1 \equiv 4k(k + 1) \equiv 0(\text{mod } 3),$$

a contradiction too. Thus $k = 1$ and $b = 3$. Therefore

$$p = 2^3 + 3^2 = 8 + 9 = 17.$$

20. Answer: 10045

The number of grid points (x, y) inside the rectangle with four corner vertices $(0, 0)$, $(11, 0)$, $(0, 2010)$ and $(11, 2010)$ is

$$(11 - 1) \times (2010 - 1) = 20090.$$

There are no grid points on the diagonal between $(0, 0)$ and $(11, 2010)$ excluding these two points, since for any point (x, y) on this segment between $(0, 0)$ and $(11, 2010)$, we have

$$y = \frac{2010x}{11}.$$

But for an integer x with $1 \leq x \leq 10$, $\frac{2010x}{11}$ is not an integer.

Hence the number of grid points (x, y) inside the triangle formed by corner points $(0, 0)$, $(0, 2010)$ and $(11, 2010)$ is

$$(11 - 1) \times (2010 - 1)/2 = 20090/2 = 10045.$$

For any grid point (x, y) inside the triangle formed by corner points $(0,0)$, $(0,2010)$ and $(11,2010)$, we have

$$1 \leq y \leq 2009, \quad 1 \leq x < \frac{11y}{2010}.$$

Thus, for any fixed positive integer y, there are the number of grid points satisfying the condition that $1 \leq x < \frac{11y}{2010}$ is

$$\left\lfloor \frac{11y}{2010} \right\rfloor,$$

as $\frac{11y}{2010}$ is not an integer when $1 \leq y \leq 2009$. Thus the number of grid points (x, y) inside the triangle formed by corner points $(0,0)$, $(0,2010)$ and $(11,2010)$ is

$$\left\lfloor \frac{11}{2010} \right\rfloor + \left\lfloor \frac{11 \times 2}{2010} \right\rfloor + \left\lfloor \frac{11 \times 3}{2010} \right\rfloor + \left\lfloor \frac{11 \times 4}{2010} \right\rfloor + \cdots + \left\lfloor \frac{11 \times 2009}{2010} \right\rfloor.$$

Therefore the answer is 10045.

21. Answer: 2007

Consider n distinct numbers where $n \geq 3$. We shall show by induction that the number of friendly pairs is always $n - 3$ for $n \geq 3$. Hence the minimal number of friendly pairs is also $n - 3$.

If $n = 3$, then there are no friendly pairs as each number is adjacent to the other two. Thus the number of friendly pairs is 0. Assume that the number of friendly pairs for any arrangement of n distinct numbers on the circle is $n - 3$ for some $n \geq 3$. Consider $n + 1$ distinct numbers on the circle. Let m be the largest number. Now consider the n numbers on circle after deleting m. The two numbers adjacent to m which originally form a friendly pair do not form a friendly pair anymore. Any other friendly pair remains a friendly pair when m is deleted. On the other hand, any friendly pair after m is deleted was originally a friendly pair. By induction hypothesis, there are $n - 3$ friendly pairs after m is deleted. Therefore, there are $(n + 1) - 3$ friendly pairs originally.

22. Answer: 12345

We are given the equation

$$\mathrm{f}(x^2 + y\mathrm{f}(z)) = x\mathrm{f}(x) + z\mathrm{f}(y). \tag{1}$$

Substituting $x = y = 0$ into (1), we get $z\mathrm{f}(0) = \mathrm{f}(0)$ for all real number z. Hence $\mathrm{f}(0) = 0$. Substituting $y = 0$ into (1), we get

$$\mathrm{f}(x^2) = x\mathrm{f}(x) \tag{2}$$

Similarly, substituting $x = 0$ in (1) we get

$$\mathrm{f}(y\mathrm{f}(z)) = z\mathrm{f}(y). \tag{3}$$

Substituting $y = 1$ into (3) we get

$$\mathrm{f}(\mathrm{f}(z)) = z\mathrm{f}(1) \tag{4}$$

for all real z. Now, using (2) and (4), we obtain

$$\mathrm{f}(x\mathrm{f}(x)) = \mathrm{f}(\mathrm{f}(x^2)) = x^2\mathrm{f}(1). \tag{5}$$

Substituting $y = z = x$ into (3) also yields

$$f(xf(x)) = xf(x). \qquad (6)$$

Comparing (5) and (6), it follows that $x^2 f(1) = xf(x)$, so that if x is non-zero, then $f(x) = cx$ for some constant c. Since $f(0) = 0$, we also have $f(x) = cx$ for all real values of x. Substituting this into (1), we have that $c(x^2 + cyz) = cx^2 + cyz$. This implies that $c^2 = c$, forcing $c = 0$ or $c = 1$. Since f is non-zero, we must have $c = 1$, so that $f(x) = x$ for all real values of x. Hence $f(12345) = 12345$.

23. Answer: 86422

The number of ways of choosing r objects from n different types of objects of which repetition is allowed is $\binom{n+r-1}{r}$. In particular, if we write r-digit numbers using n digits allowing for repetitions with the condition that the digits appear in a non-increasing order, there are $\binom{n+r-1}{r}$ ways of doing so. Grouping the given numbers into different categories, we have the following tabulation. In order to track the enumeration of these elements, the cumulative sum is also computed:

Numbers Beginning with	Digits used other than the fixed part	n	r	$\binom{n+r-1}{r}$	Cumulative
1	$1,0$	2	5	$\binom{6}{5} = 6$	6
2	$2,1,0$	3	5	$\binom{7}{5} = 21$	27
3	$3,2,1,0$	4	5	$\binom{8}{5} = 56$	83
4	$4,3,2,1,0$	5	5	$\binom{9}{5} = 126$	209
5	$5,4,3,2,1,0$	6	5	$\binom{10}{5} = 252$	461
6	$6,5,4,3,2,1,0$	7	5	$\binom{11}{5} = 462$	923
7	$7,6,5,4,3,2,1,0$	8	5	$\binom{12}{5} = 792$	1715
From 800000 to 855555	$5,4,3,2,1,0$	6	5	$\binom{10}{5} = 252$	1967
From 860000 to 863333	$3,2,1,0$	4	4	$\binom{7}{4} = 35$	2002

The next 6-digit numbers are:

$864000, 864100, 864110, 864111, 864200, 864210, 864211, 864220$. Hence, the 2010^{th} 6-digit number is 864220. Therefore, $x = 864220$ so that $\lfloor \frac{x}{10} \rfloor = 86422$.

24. Answer: 309

Let S be the set of permutations of the six integers from 1 to 6. Then $|S| = 6! = 720$. Define $P(i)$ to be the subset of S such that the digit $i+1$ follows immediately i, $i = 1, 2, 3, 4, 5$.

Then

$$\sum_i |P(i)| = \binom{5}{1}5!$$

$$\sum_{i_1<i_2} \left|P(i_1)\bigcap P(i_2)\right| = \binom{5}{2}4!$$

$$\sum_{i_1<i_2<i_3} \left|P(i_1)\bigcap P(i_2)\bigcap P(i_3)\right| = \binom{5}{3}3!$$

$$\sum_{i_1<i_2<i_3<i_4} \left|P(i_1)\bigcap P(i_2)\bigcap P(i_3)\bigcap P(i_4)\right| = \binom{5}{4}2!$$

$$\left|P(1)\bigcap P(2)\bigcap P(3)\bigcap P(4)\bigcap P(5)\right| = \binom{5}{5}1!.$$

By the Principle of Inclusion and Exclusion, the required number is

$$6! - \binom{5}{1}5! + \binom{5}{2}4! - \binom{5}{3}3! + \binom{5}{4}2! - \binom{5}{5}1! = 309.$$

25. Answer: 2011

Note that

$$A = \left(\binom{2010}{0} - \binom{2010}{-1}\right)^2 + \left(\binom{2010}{1} - \binom{2010}{0}\right)^2 + \left(\binom{2010}{2} - \binom{2010}{1}\right)^2$$

$$+\cdots+\left(\binom{2010}{1005} - \binom{2010}{1004}\right)^2 = \frac{1}{2}\sum_{k=0}^{2011}\left(\binom{2010}{k} - \binom{2010}{k-1}\right)^2.$$

Observe that

$$\sum_{k=0}^{2011}\left(\binom{2010}{k} - \binom{2010}{k-1}\right)^2$$

is the coefficient of x^{2010} in the expansion of the following expression:

$$\left(\sum_{k=0}^{2011}\left(\binom{2010}{k} - \binom{2010}{k-1}\right)x^k\right)\left(\sum_{k=-1}^{2010}\left(\binom{2010}{k} - \binom{2010}{k+1}\right)x^k\right).$$

We also have

$$\sum_{k=0}^{2011}\left(\binom{2010}{k} - \binom{2010}{k-1}\right)x^k = \sum_{k=0}^{2011}\binom{2010}{k}x^k - \sum_{k=0}^{2011}\binom{2010}{k-1}x^k$$

$$= (x+1)^{2010} - x(x+1)^{2010} = (1-x)(x+1)^{2010}$$

and

$$\sum_{k=-1}^{2010}\left(\binom{2010}{k} - \binom{2010}{k+1}\right)x^k = \sum_{k=0}^{2010}\binom{2010}{k}x^k - \frac{1}{x}\sum_{k=-1}^{2009}\binom{2010}{k+1}x^{k+1}$$

$$= (x+1)^{2010} - \frac{1}{x}(x+1)^{2010} = (1-1/x)(x+1)^{2010}.$$

The coefficient of x^{2010} in the expansion of the following expression:

$$(1-x)(x+1)^{2010}(1-1/x)(x+1)^{2010} = (2-1/x-x)(x+1)^{4020}$$

is

$$2\binom{4020}{2010} - \binom{4020}{2011} - \binom{4020}{2009} = 2\binom{4020}{2010} - 2\binom{4020}{2009}.$$

Hence

$$A = \binom{4020}{2010} - \binom{4020}{2009}.$$

Consider the inequality:

$$sA \geq \binom{4020}{2010},$$

$$s(1-2010/2011) \geq 1,$$

$$s \geq 2011.$$

Hence the answer is 2011.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Open Section, Round 2)

Saturday, 26 June 2010 **0900-1330**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

---

1. Let CD be a chord of a circle Γ_1 and AB a diameter of Γ_1 perpendicular to CD at N with $AN > NB$. A circle Γ_2 centred at C with radius CN intersects Γ_1 at points P and Q. The line PQ intersects CD at M and AC at K; and the extension of NK meets Γ_2 at L. Prove that PQ is perpendicular to AL.

2. Let a_n, b_n, $n = 1, 2, \ldots$ be two sequences of integers defined by $a_1 = 1, b_1 = 0$ and for $n \geq 1$,

$$a_{n+1} = 7a_n + 12b_n + 6$$
$$b_{n+1} = 4a_n + 7b_n + 3.$$

Prove that a_n^2 is the difference of two consecutive cubes.

3. Suppose that $a_1, \ldots, a_{15}$ are prime numbers forming an arithmetic progression with common difference $d > 0$. If $a_1 > 15$, prove that $d > 30,000$.

4. Let n be a positive integer. Find the smallest positive integer k with the property that for any colouring of the squares of a $2n \times k$ chessboard with n colours, there are 2 columns and 2 rows such that the 4 squares in their intersections have the same colour.

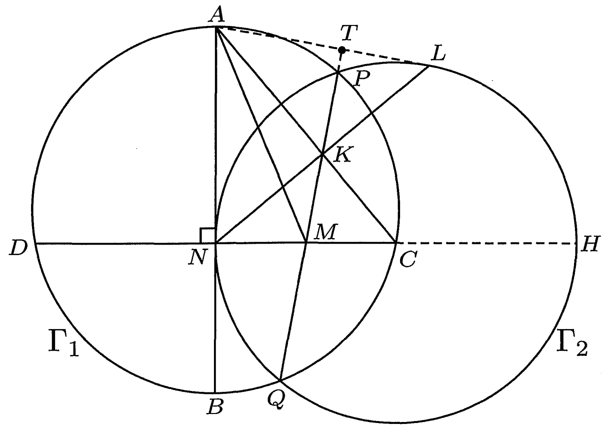
5. Let p be a prime number and let x, y, z be positive integers so that $0 < x < y < z < p$. Suppose that x^3, y^3 and z^3 have the same remainder when divided by p, show that $x^2 + y^2 + z^2$ is divisible by $x + y + z$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Open Section, Round 2 solutions)

1. Extend DC meeting Γ_2 at H. Let the radius of Γ_2 be r. Note that $DN = NC = CH = r$. Since M is of equal power with respect to Γ_1 and Γ_2. Thus $MN \cdot MH = MC \cdot MD$. That is $MN(MC + r) = MC(MN + r)$ giving $MN = MC$. Thus M is the midpoint of NC.



As K lies on the radical axis of Γ_1 and Γ_2, the points C, N, A, L are concyclic. Thus $\angle ALC = \angle ANC = 90°$ so that AL is tangent to Γ_2. It follows that AC is perpendicular to NL at K, and hence $MN = MC = MK$.

Now let PQ intersect AL at T. We have $\angle TAK = \angle KNM = \angle NKM = \angle LKT$ and similarly $\angle TLK = \angle AKT$. Consequently, $2\angle KTL = 2(\angle TAK + \angle AKT) = \angle TAK + \angle AKT + \angle LKT + \angle TLK = 180°$, which means $\angle KTL = 90°$.

2. First we shall prove that a_n^2 is the difference of two consecutive cubes. To do so, we shall prove by induction that $a_n^2 = (b_n + 1)^3 - b_n^3$. When $n = 1$, this is true. Suppose for $n \geq 1$, this is true. We have

$$\begin{aligned}
(b_{n+1} + 1)^3 - b_{n+1}^3 &= 3b_{n+1}^2 + 3b_{n+1} + 1 \\
&= 3(4a_n + 7b_n + 3)^2 + 3(4a_n + 7b_n + 3) + 1 \\
&= 48a_n^2 + 147b_n^2 + 168a_nb_n + 84a_n + 147b_n + 37 \\
&= (7a_n + 12b_n + 6)^2 + (3b_n^2 + 3b_n + 1) - a_n^2 \\
&= a_{n+1}^2 + (b_n + 1)^3 - b_n^3 - a_n^2 = a_{n+1}^2
\end{aligned}$$

where the last equality is by induction hypothesis.

3. Lemma: Suppose p is prime and $a_1, \ldots, a_p$ are primes forming an A.P. with common difference d. If $a_1 > p$, we claim that $p \mid d$.

Proof: Since p is prime and every a_i is a prime $> p$, p does not divide a_i for any i. By the pigeonhole principle, there exist $1 \le i < j \le p$ so that $a_i \equiv a_j \pmod{p}$. Now $a_j - a_i = (j-i)d$, and p does not divide $j - i$. So p must divide d.

Apply the Lemma to the sequences $a_1, \ldots, a_k$ for $k = 2, 3, 5, 7, 11$ and 13. Then all such k's are factors of d. So $d > 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 > 30,000$.

4. The answer is $2n^2 - n + 1$.

Consider an n-colouring of the $2n \times k$ chessboard. A *vertical-pair* is a pair of squares in the same column that are coloured the same. In every column there are at least n vertical-pairs. Let P be the total number of vertical-pairs and P_i be the number of vertical-pairs with colour i. Then $P = P_1 + \cdots + P_n \ge nk$. Thus there is colour i with $P_i \ge k$. There are $\binom{2n}{2} = 2n^2 - n$ pairs of rows. Thus if $k \ge 2n^2 - n + 1$, there is a pair of rows that contains two vertical-pairs with colour i.

Next for $k = 2n^2 - n$, exhibit an n-colouring where no such sets of 4 squares exists. Note that it suffices to find such an n-colouring for the $2n \times (2n-1)$ board. We can then rotate the colours to obtain n of these boards which can then be put together to obtain the requiring n-colouring of the $2n \times (2n^2 - n)$ board. For each $i = 1, 2, \ldots, 2n-1$, let $A_i = \{(i, 2n-1+i), (i+1, 2n-2+i), \ldots, (n-1+i, n+i)\}$, where $2n+k$, $k > 0$, is taken to be k. Note that the pairs in each A_i give a partition of $\{1, 2, \ldots, 2n\}$. Moreover, each pair of elements appears in exactly one A_i. Now colour the squares of column i using n colours so that the two squares in each pair of A_i receive the same colour and the colours the $2n$ pairs are mutually distinct. This gives an n-colouring of the $2n \times 2n - 1$ board with the required property and we are done.

5. It is given that

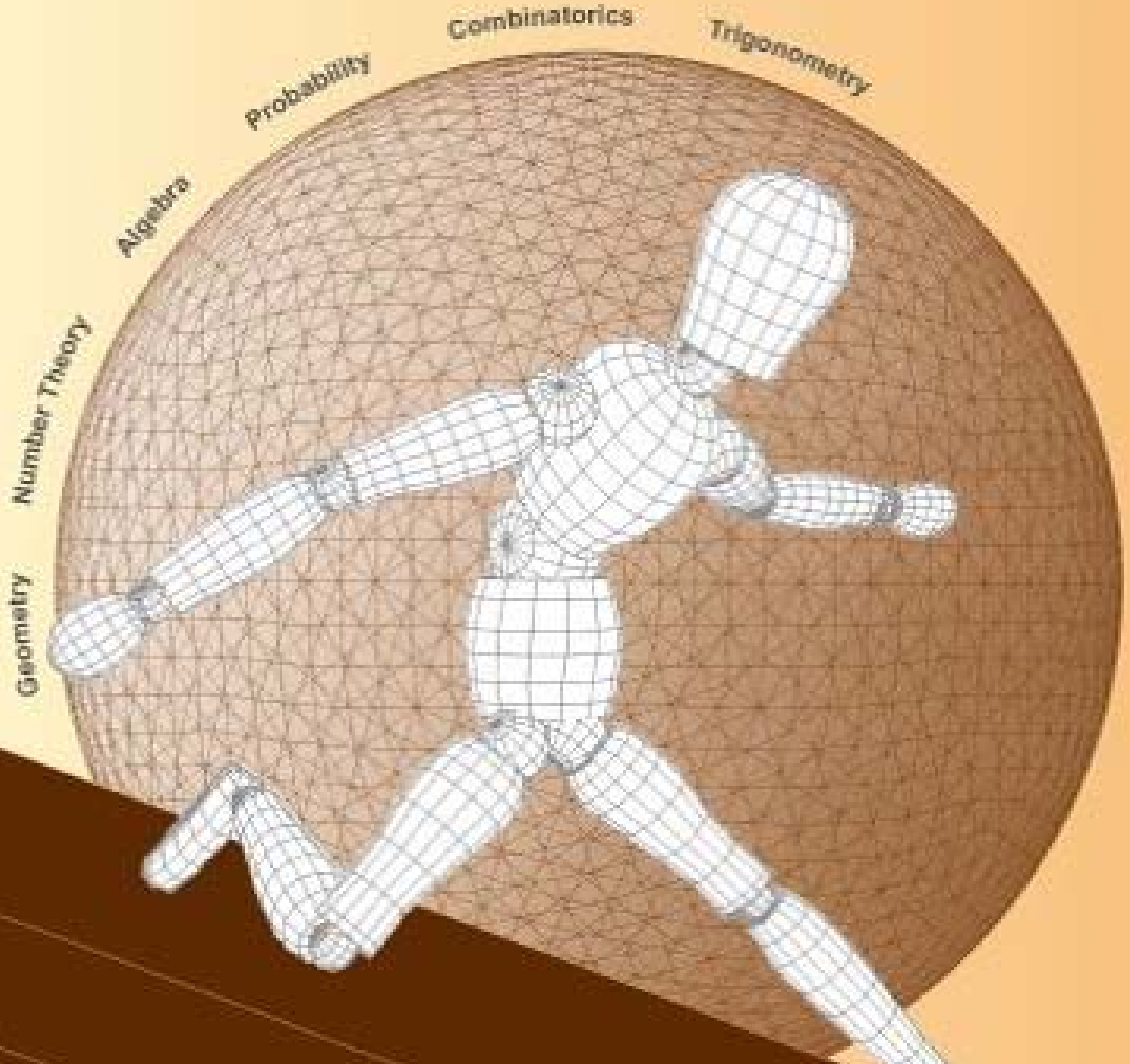
$$p \mid x^3 - y^3 = (x-y)(x^2 + xy + y^2).$$

Moreover, $-p < x - y < 0$ and so $p \nmid x - y$. Thus $p \mid x^2 + xy + y^2$. Similarly, $p \mid y^2 + yz + z^2$ and $p \mid x^2 + xz + z^2$. Hence

$$p \mid x^2 + xy - yz - z^2 = (x-z)(x+y+z).$$

Since $p \nmid x - z$, $p \mid x + y + z$. Note that $0 < x + y + z < 3p$. So $x + y + z = p$ or $2p$. We will show that $p \mid x^2 + y^2 + z^2$. Assuming this for the moment, the proof is complete if $x + y + z = p$. If $x + y + z = 2p$, then $x + y + z$ is even and hence $x^2 + y^2 + z^2$ is even. So both 2 and p divide $x^2 + y^2 + z^2$. Moreover, $p > 2$ and so 2 and p are relatively prime. Thus $2p$ divides $x^2 + y^2 + z^2$ and the proof is also complete in this case.

SINGAPORE MATHEMATICAL OLYMPIADS 2011



Tay Tiong Seng
To Wing Keung
Toh Tin Lam
Wang Fei

Published by Singapore Mathematical Society

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Junior Section)

Tuesday, 31 May 2011 0930–1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer on the answer sheet and shade the appropriate bubbles below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Calculate the following sum:

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots+\frac{10}{2^{10}}.$$

(A) $\frac{503}{256}$; (B) $\frac{505}{256}$; (C) $\frac{507}{256}$; (D) $\frac{509}{256}$; (E) None of the above.

2. It is known that the roots of the equation

$$x^5+3x^4-4044118x^3-12132362x^2-12132363x-2011^2=0$$

are all integers. How many distinct roots does the equation have?

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

3. A fair dice is thrown three times. The results of the first, second and third throw are recorded as x, y and z, respectively. Suppose $x+y=z$. What is the probability that at least one of x, y and z is 2?

(A) $\frac{1}{12}$; (B) $\frac{3}{8}$; (C) $\frac{8}{15}$; (D) $\frac{1}{3}$; (E) $\frac{7}{13}$.

4. Let

$$x=1\underbrace{000\cdots000}_{2011\text{ times}}1\underbrace{000\cdots000}_{2012\text{ times}}50.$$

Which of the following is a perfect square?

(A) $x-75$; (B) $x-25$; (C) x; (D) $x+25$; (E) $x+75$.

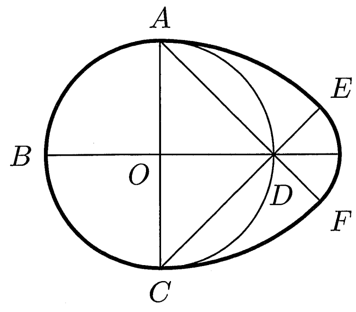
5. Suppose $N_1, N_2, \ldots, N_{2011}$ are positive integers. Let

$$X=(N_1+N_2+\cdots+N_{2010})(N_2+N_3+\cdots+N_{2011}),$$
$$Y=(N_1+N_2+\cdots+N_{2011})(N_2+N_3+\cdots+N_{2010}).$$

Which one of the following relationships always holds?

(A) $X=Y$; (B) $X>Y$; (C) $X<Y$; (D) $X-N_1<Y-N_{2011}$; (E) None of the above.

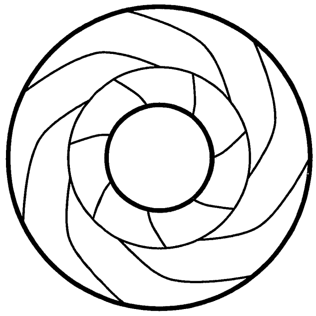
6. Consider the following egg shaped curve. $ABCD$ is a circle of radius 1 centred at O. The arc $\overset{\frown}{AE}$ is centred at C, $\overset{\frown}{CF}$ is centred at A and $\overset{\frown}{EF}$ is centred at D.



What is the area of the region enclosed by the egg shaped curve?

(A) $(3-\sqrt{2})\pi-1$; (B) $(3-\sqrt{2})\pi$; (C) $(3+\sqrt{2})\pi+1$; (D) $(3-2\sqrt{2})\pi$; (E) $(3-2\sqrt{2})\pi+1$.

7. The following annulus is cut into 14 regions. Each region is painted with one colour. What is the minimum number of colours needed to paint the annulus so that any no two adjacent regions share the same colours?



(A) 3; (B) 4; (C) 5; (D) 6; (E) 7.

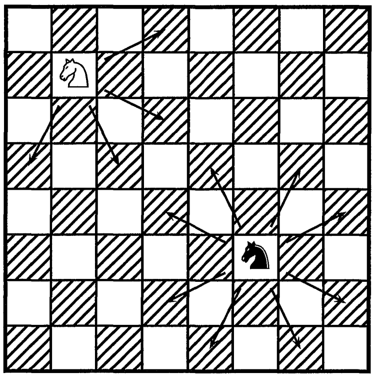
8. Let $n = (2^4 - 1)(3^6 - 1)(5^{10} - 1)(7^{12} - 1)$. Which of the following statements is true?

(A) n is divisible by $5, 7$ and 11 but not 13; (B) n is divisible by $5, 7$ and 13 but not 11;

(C) n is divisible by $5, 11$ and 13 but not 7; (D) n is divisible by $7, 11$ and 13 but not 5;

(E) None of the above.

9. How many ways can you place a White Knight and a Black Knight on an 8×8 chessboard such that they do not attack each other?



(A) 1680; (B) 1712; (C) 3696; (D) 3760; (E) None of the above.

10. In the set $\{1, 6, 7, 9\}$, which of the numbers appear as the last digit of n^n for infinitely many positive integers n?

    (A) $1, 6, 7$ only; (B) $1, 6, 9$ only; (C) $1, 7, 9$ only; (D) $6, 7, 9$ only; (E) $1, 6, 7, 9$.

Short Questions

11. Suppose $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = \sqrt{2}$ and $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = 0$. Find

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}.$$

12. Suppose $x = \dfrac{13}{\sqrt{19 + 8\sqrt{3}}}$. Find the exact value of

$$\frac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15}.$$

13. Let $a_1 = 3$, and define $a_{n+1} = \dfrac{\sqrt{3}\, a_n - 1}{a_n + \sqrt{3}}$ for all positive integers n. Find a_{2011}.

14. Let a, b, c be positive real numbers such that

$$\begin{cases} a^2 + ab + b^2 = 25, \\ b^2 + bc + c^2 = 49, \\ c^2 + ca + a^2 = 64. \end{cases}$$

    Find $(a + b + c)^2$.

15. Let $P(x)$ be a polynomial of degree 2010. Suppose $P(n) = \dfrac{n}{1+n}$ for all $n = 0, 1, 2, \ldots, 2010$. Find $P(2012)$.

16. Let $\lfloor x \rfloor$ be the greatest integer smaller than or equal to x. How many solutions are there to the equation $x^3 - \lfloor x^3 \rfloor = (x - \lfloor x \rfloor)^3$ on the interval $[1, 20]$?

17. Let n be the smallest positive integer such that the sum of its digits is 2011. How many digits does n have?

18. Find the largest positive integer n such that $n + 10$ is a divisor of $n^3 + 2011$.

19. Let a, b, c, d be real numbers such that

$$\begin{cases} a^2 + b^2 + 2a - 4b + 4 = 0, \\ c^2 + d^2 - 4c + 4d + 4 = 0. \end{cases}$$

Let m and M be the minimum and the maximum values of $(a-c)^2 + (b-d)^2$, respectively. What is $m \times M$?

20. Suppose $x_1, x_2, \ldots, x_{2011}$ are positive integers satisfying

$$x_1 + x_2 + \cdots + x_{2011} = x_1 x_2 \cdots x_{2011}.$$

Find the maximum value of $x_1 + x_2 + \cdots + x_{2011}$.

21. Suppose that a function $M(n)$, where n is a positive integer, is defined by

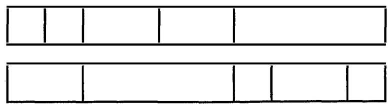
$$M(n) = \begin{cases} n - 10 & \text{if } n > 100, \\ M(M(n+11)) & \text{if } n \leq 100. \end{cases}$$

How many solutions does the equation $M(n) = 91$ have?

22. For each positive integer n, define $A_n = \dfrac{20^n + 11^n}{n!}$, where $n! = 1 \times 2 \times \cdots \times n$. Find the value of n that maximizes A_n.

23. Find the number of ways to pave a 1×10 block with tiles of sizes 1×1, 1×2 and 1×4, assuming tiles of the same size are indistinguishable. (For example, the following are two

distinct ways of using two tiles of size 1×1, two tiles of size 1×2 and one tile of size 1×4. It is not necessary to use all the three kinds of tiles.)



24. A 4×4 Sudoku grid is filled with digits so that each column, each row, and each of the four 2×2 sub-grids that composes the grid contains all of the digits from 1 to 4. For example,

4	3	1	2
2	1	3	4
1	2	4	3
3	4	2	1

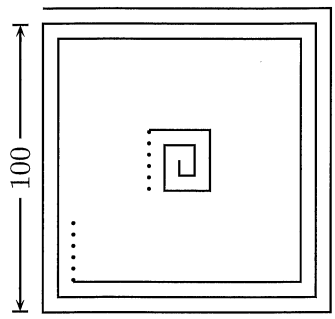
Find the total number of possible 4×4 Sudoku grids.

25. If the 13th of any particular month falls on a Friday, we call it *Friday the 13th.* It is known that Friday the 13th occurs at least once every calendar year. If the longest interval between two consecutive occurrences of Friday the 13th is x months, find x.

26. How many ways are there to put 7 identical apples into 4 identical packages so that each package has at least one apple?

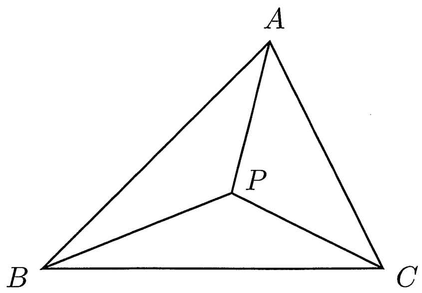
27. At a fun fair, coupons can be used to purchase food. Each coupon is worth \$5, \$8 or \$12. For example, for a \$15 purchase you can use three coupons of \$5, or use one coupon of \$5 and one coupon of \$8 and pay \$2 by cash. Suppose the prices in the fun fair are all whole dollars. What is the largest amount that you cannot purchase using only coupons?

28. Find the length of the spirangle in the following diagram, where the gap between adjacent parallel lines is 1 unit.



29. There are two fair dice and their sides are positive integers $a_1, \ldots, a_6$ and $b_1, \ldots, b_6$, respectively. After throwing them, the probability of getting a sum of $2, 3, 4, \ldots, 12$ respectively is the same as that of throwing two normal fair dice. Suppose that $a_1 + \cdots + a_6 < b_1 + \cdots + b_6$. What is $a_1 + \cdots + a_6$?

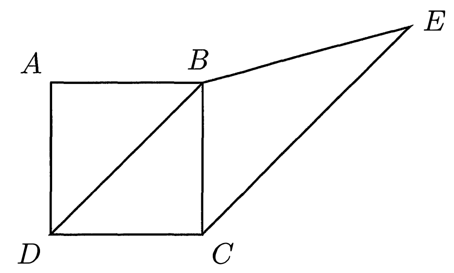
30. Consider a triangle ABC, where $AB = 20$, $BC = 25$ and $CA = 17$. P is a point on the plane. What is the minimum value of $2 \times PA + 3 \times PB + 5 \times PC$?



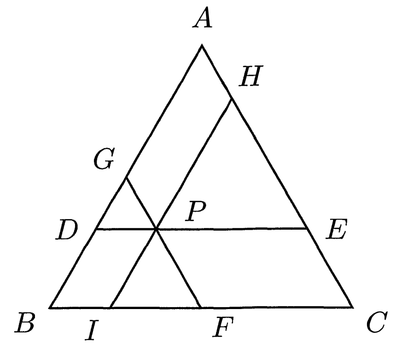
31. Given an equilateral triangle, what is the ratio of the area of its circumscribed circle to the area of its inscribed circle?

32. Let A and B be points that lie on the parabola $y = x^2$ such that both are at a distance of $8\sqrt{2}$ units from the line $y = -x - 4$. Find the square of the distance between A and B.

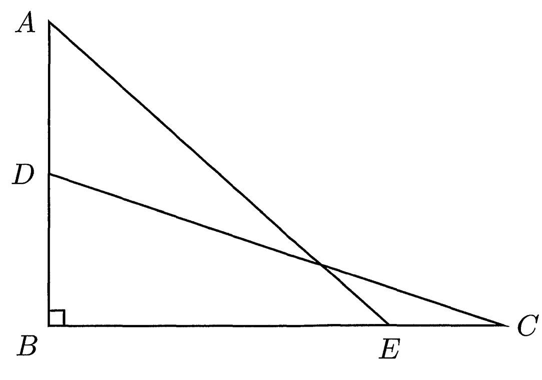
33. In the following diagram, $ABCD$ is a square, $BD \parallel CE$ and $BE = BD$. Let $\angle E = x°$. Find x.



34. Consider an equilateral triangle ABC, where $AB = BC = CA = 2011$. Let P be a point inside $\triangle ABC$. Draw line segments passing through P such that $DE \parallel BC$, $FG \parallel CA$ and $HI \parallel AB$. Suppose $DE : FG : HI = 8 : 7 : 10$. Find $DE + FG + HI$.



35. In the following diagram, $AB \perp BC$. D and E are points on segments AB and BC respectively, such that $BA + AE = BD + DC$. It is known that $AD = 2$, $BE = 3$ and $EC = 4$. Find $BA + AE$.



Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Junior Section)

Multiple Choice Questions

1. Answer: (D)

   Note that $\frac{n}{2^n} = \frac{n+1}{2^{n-1}} - \frac{n+2}{2^n}$. Then

$$\begin{aligned}\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{10}{2^{10}} &= \left(\frac{2}{2^0}-\frac{3}{2^1}\right)+\left(\frac{3}{2^1}-\frac{4}{2^2}\right)+\left(\frac{4}{2^2}-\frac{5}{2^3}\right)+\cdots+\left(\frac{11}{2^9}-\frac{12}{2^{10}}\right)\\ &= \frac{2}{2^0}-\frac{12}{2^{10}}=\frac{509}{256}.\end{aligned}$$

2. Answer: (C).

   Let the roots be $n_1 \le n_2 \le n_3 \le n_4 \le n_5$. Then the polynomial can be factorized as

   $(x-n_1)(x-n_2)(x-n_3)(x-n_4)(x-n_5) = x^5-(n_1+n_2+n_3+n_4+n_5)x^4+\cdots-n_1n_2n_3n_4n_5$.

   Compare the coefficients: $n_1+n_2+n_3+n_4+n_5=-3$ and $n_1n_2n_3n_4n_5=2011^2$. Then

$$n_1=-2011, \quad n_2=n_3=n_4=-1, \quad n_5=2011.$$

3. Answer: (C).

   If $z=2$, then $(x,y)=(1,1)$.

   If $z=3$, then $(x,y)=(1,2),(2,1)$.

   If $z=4$, then $(x,y)=(1,3),(2,2),(3,1)$.

   If $z=5$, then $(x,y)=(1,4),(2,3),(3,2),(4,1)$.

   If $z=6$, then $(x,y)=(1,5),(2,4),(3,3),(4,2),(5,1)$.

   Out if these 15 cases, 8 of them contain at least one 2. Hence, the required probability is $\frac{8}{15}$.

4. Answer: (B).

Note that

$$x = (10^{1012} + 1) \times 10^{2014} + 50 = 10^{4026} + 10^{2014} + 50$$
$$= (10^{2013})^2 + 2 \times 10^{2013} \times 5 + 50 = (10^{2013} + 5)^2 + 25.$$

So $x - 25$ is a perfect square.

5. Answer: (B).

Let $K = N_2 + \cdots + N_{2010}$. Then $X = (N_1 + K)(K + N_{2011})$ and $Y = (N_1 + K + N_{2011})K$.

$$X - Y = (N_1K + K^2 + N_1N_{2011} + KN_{2011}) - (N_1K + K^2 + N_{2011}K) = N_1N_{2011} > 0.$$

6. Answer: (A).

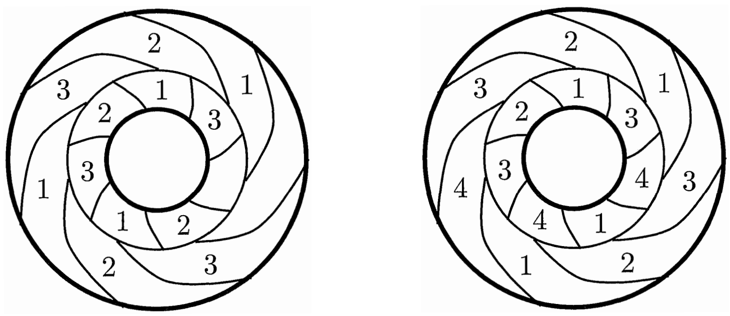
The area enclosed by AD, DE and $\overset{\frown}{AE}$ is $\dfrac{\pi(2^2)}{8} - 1 = \dfrac{\pi}{2} - 1$.

The area of the wedge EDF is $\dfrac{\pi(2-\sqrt{2})^2}{4} = \left(\dfrac{3}{2} - \sqrt{2}\right)\pi$.

So the area of the egg is: $\dfrac{\pi}{2} + 1 + 2 \times \left(\dfrac{\pi}{2} - 1\right) + \left(\dfrac{3}{2} - \sqrt{2}\right)\pi = (3 - \sqrt{2})\pi - 1$.

7. Answer: (B).

The left shows that 3 colours are not enough. The right is a painting using 4 colours.



8. Answer: (E).

Since $5 \mid (2^4 - 1)$, $7 \mid (3^6 - 1)$, $11 \mid (5^{10} - 1)$, $13 \mid (7^{12} - 1)$, n is divisible by $5, 7, 11$ and 13.

9. Answer: (C).

We consider the position of the Black Knight. The number of positions being attacked by the White Knight can be counted.

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

There are $16 \times 8 + 16 \times 6 + 20 \times 4 + 8 \times 3 + 4 \times 2 = 336$ cases. Hence, the total number of cases that the Kights do not attack each other is $64 \times 63 - 336 = 3696$.

10. Answer: (E).

Note that the power of any positive integer n with last digit 1 or 6 is 1 or 6 respectively.

If the last digit of n is 9, then $n^2 \equiv 1 \pmod{10}$, and $n^n \equiv n^{10k+9} \equiv -1 \equiv 9 \pmod{10}$.

If the last digit of n is 7, then $n^4 \equiv 1 \pmod{10}$. Suppose the second last digit of n is odd. Then $n^n \equiv n^{20k+17} \equiv 7 \pmod{10}$.

Short Questions

11. Answer: 2.

$$2 = \left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+2\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ca}\right) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}+2\frac{xyz}{abc}\left(\frac{c}{z}+\frac{a}{x}+\frac{b}{y}\right).$$

So $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=2.$

12. Answer: 5.

Note that $x = \dfrac{1}{13\sqrt{(4+\sqrt{3})^2}} = \dfrac{13(4-\sqrt{3})}{(4-\sqrt{3})(4+\sqrt{3})} = 4-\sqrt{3}$. So $(x-4)^2 = 3$. That is,

$$x^2 - 8x + 15 = 2.$$

It follows that

$$\begin{aligned}\frac{x^4-6x^3-2x^2+18x+23}{x^2-8x+15} &= x^2+2x-1+\frac{38-20x}{x^2-8x+15}\\ &= x^2+2x-1+\frac{38-20x}{2}\\ &= x^2-8x+18\\ &= 2+3=5.\end{aligned}$$

13. Answer: 3.

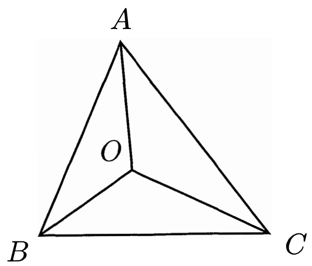
Let $f(x) = \dfrac{\sqrt{3}x - 1}{x + \sqrt{3}}$. Then $f(f(x)) = \dfrac{\sqrt{3}\,\frac{\sqrt{3}-1}{x+\sqrt{3}} - 1}{\frac{\sqrt{3}x-1}{x+\sqrt{3}} + \sqrt{3}} = \dfrac{3x - \sqrt{3} - x - \sqrt{3}}{\sqrt{3}x - 1 + \sqrt{3}x + 3} = \dfrac{x - \sqrt{3}}{\sqrt{3}x + 1}$.

$f(f(f(x))) = \dfrac{\frac{\sqrt{3}x-1}{x+\sqrt{3}} - \sqrt{3}}{\sqrt{3}\,\frac{\sqrt{3}x-1}{x+\sqrt{3}} + 1} = \dfrac{\sqrt{3}x - 1 - \sqrt{3}x - 3}{3x - \sqrt{3} + x + \sqrt{3}} = -\dfrac{1}{x}$. So $f(f(f(f(f(f(x)))))) = x$.

Since $2010 = 6 \times 335$, $a_{2011} = \underbrace{f(f(f \cdots f(f}_{2010 \text{ copies}}(3)) \cdots)) = 3$.

14. Answer: 129.

Consider the following picture, where $\angle AOB = \angle BOC = \angle COA = 120^\circ$, $OA = a$, $OB = b$ and $OC = c$.



Then $|BC| = 5$, $|CA| = 7$ and $|AB| = 8$. The area of the triangle ABC is

$$\sqrt{10(10-5)(10-7)(10-8)} = 10\sqrt{3}.$$

Then $\dfrac{1}{2}\dfrac{\sqrt{3}}{2}(ab + bc + ca) = 10\sqrt{3}$. So $ab + bc + ca = 40$.

$$2(a+b+c)^2 = (a^2 + ab + b^2) + (b^2 + bc + c^2) + (c^2 + ca + a^2) + 3(ab + bc + ca) = 258.$$

Thus, $(a + b + c)^2 = 129$.

15. Answer: 0.

Define $Q(x) = (1 + x)P(x) - x$. Then $Q(x)$ is a polynomial of degree 2011. Since $Q(0) = Q(1) = Q(2) = \cdots = Q(2010) = 0$, we can write, for some constant A,

$$Q(x) = Ax(x-1)(x-2)\cdots(x-2010).$$

$1 = Q(-1) = A(-1)(-2)(-3)\cdots(-2011) = -A \cdot 2011!$. Then $Q(2012) = A \cdot 2012! = -2012$, and $P(2012) = \dfrac{Q(2012) + 2012}{2013} = 0$.

16. Answer: 9241.

Let $n = \lfloor x \rfloor$, $\{x\} = x - n$. The equation becomes $(n + \{x\})^3 - \{x\}^3 = \lfloor (n + \{x\})^3 \rfloor$. Then

$$3n\{x\}(n + \{x\}) = \lfloor 3n\{x\}(n + \{x\}) + \{x\}^3 \rfloor .$$

The right-hand side is an integer. The above holds if and only if $3n\{x\}(n+\{x\})$ is an integer.

Note that $0 \le 3n\{x\}(n+\{x\}) < 3n(n+1)$. There are exactly $3n(n+1)$ solutions in $[n, n+1)$, $n = 1, 2, \ldots$. So on $[1, 20]$, the total number of solutions is

$$\begin{aligned} & 3(1 \times 2 + 2 \times 3 + \cdots 20 \times 21) + 1 \\ = & (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (21^3 - 20^3) - 20 + 1 \\ = & 21^3 - 20 = 9241. \end{aligned}$$

17. Answer: 224.

For smallest possible n, we need to have 9 as the digits of n as many as possible. So n is the integer whose first digit is $2011 - 223 \times 9 = 4$ and followed by 223 9's.

18. Answer: 1001.

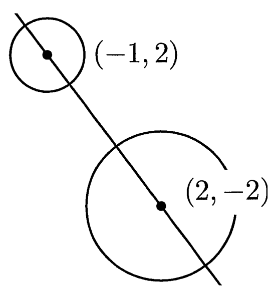
$\dfrac{n^3 + 2011}{n + 10} = n^2 - 10n + 100 + \dfrac{1011}{n + 10}$. This is an integer if and only if $(n + 10) \mid 1011$. The maximum value of n is $1011 - 10 = 1001$.

19. Answer: 16.

Complete the squares: $(a + 1)^2 + (b - 2)^2 = 1$ and $(c - 2)^2 + (d + 2)^2 = 2^2$. Each of them represents a circle.

The distance between the two centres is $\sqrt{(2 - (-1))^2 + (-2 - 2)^2} = 5$.

So $m = 5 - 1 - 2 = 2$ and $M = 5 + 1 + 2 = 8$. Thus, $m \times M = 16$.



20. Answer: 4022.

Suppose $x_1 = x_2 = \cdots = x_k = 1 < 2 \le x_{k+1} \le \cdots \le x_{2011}$. Let $M = x_1 \cdots x_{2011}$. Then

$$\begin{aligned}
M &= x_{k+1}x_{k+2}\cdots x_{2010}x_{2011} \\
&= (x_{k+1}-1)x_{k+2}\cdots x_{2010}x_{2011} + x_{k+2}\cdots x_{2010}x_{2011} \\
&\ge (x_{k+1}-1)2 + x_{k+2}\cdots x_{2010}x_{2011} \\
&\ge \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\ge (x_{k+1}-1)2 + \cdots + (x_{2009}-1)2 + x_{2010}x_{2011} \\
&\ge (x_{k+1}-1)2 + \cdots + (x_{2009}-1)2 + (x_{2010}-1)2 + (x_{2011}-1)2 \\
&= 2(x_{k+1} + \cdots + x_{2011} - (2011-k)) \\
&= 2(M - 2011).
\end{aligned}$$

Therefore, $M \le 4022$. On the other hand, $(1, 1, \ldots, 1, 2, 2011)$ is a solution to the equation. So the maximum value is 4022.

21. Answer: 101.

If $n \ge 102$, then $M(n) = n - 10 \ge 92$.

$M(91) = M(M(102)) = M(92) = M(M(103)) = M(93) = \cdots = M(101) = 91$.

For each $k = 1, \ldots, 10$, $M(80+k) = M(M(91+k)) = M(91) = 91$, and thus

$$\begin{gathered}
M(70+k) = M(M(81+k)) = M(91) = 91, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
M(k) = M(M(11+k)) = M(91) = 91.
\end{gathered}$$

Hence, all integers from 1 to 101 are solutions to $M(n) = 91$.

22. Answer: 19.

$$\frac{A_{n+1}}{A_n} = \frac{1}{n+1}\,\frac{20^{n+1}+11^{n+1}}{20^n+11^n} = \frac{20 + 11\cdot(\frac{11}{20})^n}{(n+1)(1+(\frac{11}{20})^n)}.$$

Then $A_{n+1} < A_n$ if $n > 10 + \dfrac{9}{1+(\frac{11}{20})^n}$; and $A_{n+1} > A_n$ if $n < 10 + \dfrac{9}{1+(\frac{11}{20})^n}$.

Note that $10 + \dfrac{9}{1+(\frac{11}{20})^n} < 10 + 9 = 19$. So $n \ge 19$ implies $A_n > A_{n+1}$.

If $10 \le n \le 18$, then $n \le 10+8 < 10+\dfrac{9}{1+(\frac{11}{20})^n}$; if $n < 10$, then $n < 10+\dfrac{9}{1+(\frac{11}{20})^n}$. Hence, $n \le 18$ implies $A_n < A_{n+1}$.

23. Answer: 169.

Let a_n be the number of ways to pave a block of $1 \times n$. Then $a_n = a_{n-1} + a_{n-2} + a_{n-4}$ with initial conditions $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_4 = 6$. Then

$$a_5 = a_4 + a_3 + a_1 = 10, \qquad a_6 = a_5 + a_4 + a_2 = 18,$$
$$a_7 = a_6 + a_5 + a_3 = 31, \qquad a_8 = a_7 + a_6 + a_4 = 55,$$
$$a_9 = a_8 + a_7 + a_5 = 96, \qquad a_{10} = a_9 + a_8 + a_6 = 169.$$

24. Answer: 288.

Consider the grid below. Suppose the left-top 2×2 sub-grid is filled in with $1, 2, 3, 4$.

If x, y, z, w are all distinct, then there are no other numbers to place in a; if $\{x, y\} = \{z, w\}$, then x', y', z, w are all distinct, and there are no other numbers for a'.

Note that $\{x, x'\} = \{1, 2\}$, $\{y, y'\} = \{3, 4\}$, $\{z, z'\} = \{2, 4\}$ and $\{w, w'\} = \{1, 3\}$. Among these $2^4 = 16$ choices, 4 of them are impossible — $\{x, y\} = \{z, w\} = \{1, 4\}$ or $\{2, 3\}$, $\{x, y\} = \{1, 4\}$ and $\{z, w\} = \{2, 3\}$, $\{x, y\} = \{2, 3\}$ and $\{z, w\} = \{1, 4\}$.

For each of the remaining 12 cases, x', y', z', w' are uniquely determined, so is the right-bottom sub-grid:

$$\{a\} = \{1, 2, 3, 4\} - \{x, y\} \cup \{z, w\},$$
$$\{b\} = \{1, 2, 3, 4\} - \{x, y\} \cup \{z', w'\},$$
$$\{a'\} = \{1, 2, 3, 4\} - \{x', y'\} \cup \{z, w\},$$
$$\{b'\} = \{1, 2, 3, 4\} - \{x', y'\} \cup \{z', w'\}.$$

Recall that that there are $4! = 24$ permutations in the left-top grid. Hence, there are $24 \times 12 = 288$ solutions.

1	2	x	x'
3	4	y	y'
z	w	a	a'
z'	w'	b	b'

25. Answer: 14.

If 13th of January falls on a particular day, represented by 0, then the 13th of February falls 3 days later, represented by $0 + 31 \equiv 3 \pmod 7$.

Case 1: The consecutive two years are non-leap years.

Jan	Feb	Mar	Apr	Mar	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0	3	3	6	1	4	$\boxed{6}$	2	5	0	3	5
1	4	4	0	2	5	0	3	$\boxed{6}$	1	4	6

Case 2: The first year is a leap year.

Jan	Feb	Mar	Apr	Mar	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0	3	4	0	2	5	$\boxed{0}$	3	6	1	4	6
2	5	5	1	3	6	1	4	$\boxed{0}$	2	5	0

Case 3: The second year is a leap year.

Jan	Feb	Mar	Apr	Mar	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0	3	3	6	1	4	$\boxed{6}$	2	5	0	3	5
1	4	5	1	3	$\boxed{6}$	1	4	0	2	5	0

From these tables we see that the answer is 14. The longest time period occurs when the Friday of 13th falls in July of the first year and in September of the second year, while the second year is not a leap year.

26. Answer: 350.

By considering the numbers of apples in the packages, there are 3 cases:

1) $(4,1,1,1)$. $\binom{7}{4} = \frac{7\times6\times5\times4}{4\times3\times2\times1} = 35.$

2) $(3,2,1,1)$. $\binom{7}{3}\binom{4}{2} = \frac{7\times6\times5}{3\times2\times1} = \times\frac{4\times3}{2\times1} = 35\times6 = 210.$

3) $(2,2,2,1)$. $\frac{1}{3!}\binom{7}{2}\binom{5}{2}\binom{3}{2} = \frac{1}{6}\times\frac{7\times6}{2\times1}\times\frac{5\times4}{2\times1}\times\frac{3\times2}{2\times1} = 105.$

So the total number of ways is $35+210+105 = 350$.

27. Answer: 19.

Note that $8+12 = 20$, $5+8+8 = 21$, $5+5+12 = 22$, $5+5+5+8 = 23$, $8+8+8 = 24$. If $n \geq 25$, write $n = 5k+m$ where $20 \leq m \leq 24$ and k is a positive integer. So any amount ≥ 25 can be paid exactly using coupons.

However, 19 cannot be paid exactly using these three types of coupons.

28. Answer: 10301.

The broken line is constructed using "L", with lengths $2,4,6,\ldots,200$. The last "L" is $100+101 = 201$. Then the total length is $2(1+2+3+\cdots+100)+201 = 10301$.

29. Answer: 15.

Let $P(x) = x^{a_1} + \cdots + x^{a_6}$ and $Q(x) = x^{b_1} + \cdots + x^{b_6}$. Then

$$P(x)Q(x) = (x + x^2 + \cdots + x^6)^2 = x^2(1+x)^2(1+x+x^2)^2(1-x+x^2)^2.$$

Note that $P(0) = Q(0) = 0$ and $P(1) = Q(1) = 6$, $x(1+x)(1+x+x^2)$ is a common divisor of $P(x)$ and $Q(x)$.

Since they are not normal dice,

$$P(x) = x(1+x)(1+x+x^2) = x + 2x^2 + 2x^3 + x^4,$$
$$Q(x) = x(1+x)(1+x+x^2)(1-x+x^2) = x + x^3 + x^4 + x^5 + x^6 + x^8.$$

So the numbers of the first dice are $1, 2, 2, 3, 3, 4$ and that of the second dice are $1, 3, 4, 5, 6, 8$.

Then $a_1 + \cdots + a_6 = 1 + 2 + 2 + 3 + 3 + 4 = 15$.

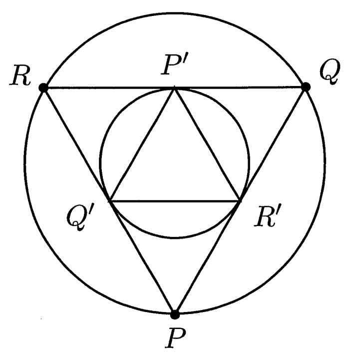
30. Answer: 109.

$2 \cdot PA + 3 \cdot PB + 5 \cdot PC = 2(PA + PC) + 3(PB + PC) \geq 2 \cdot AC + 3 \cdot BC = 2 \cdot 17 + 3 \cdot 25 = 109$.

The equality holds if and only if $P = C$.

31. Answer: 4.

Given an equilateral triangle PQR. Let C_1 be its circumscribed circle and C_2 its inscribed circle. Suppose QR, RP, PQ are tangent to C_2 at P', Q', R', respectively. The area of triangle PQR is 4 times the area of triangle $P'Q'R'$. So the area of C_1 is also 4 times the area of C_2.



32. Answer: 98.

Since $y = x^2$ and $y = -x - 4$ do not intersect, A and B must lie on a line parallel to $y = -x - 4$, namely, $y = -x + c$. The distance from $(0, -4)$ to $y = -x + c$ is

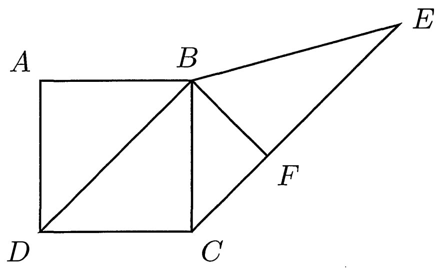
$$8\sqrt{2} = \frac{|(0) + (-4) - c|}{\sqrt{1+1}}.$$

So $c = 12$ or $c = -20$ (ignored). Substitute $y = -x + 12$ into the parabola: $x^2 = -x + 12 \Rightarrow x = 3, -4$. So A is $(3, 9)$ and B is $(-4, 16)$. Then

$$|AB|^2 = (3 - (-4))^2 + (9 - 16)^2 = 98.$$

33. Answer: 30.

Draw $BF \perp CE$, where F is on CE. If $AB = 1$, then $BF = \frac{\sqrt{2}}{2}$ and $BE = \sqrt{2}$. Thus $\angle E = 30°$.



34. Answer: 4022.

Set $DP = GP = a$, $IP = FP = b$, $EP = HP = c$. Then

$$DE + FG + HI = (a + c) + (a + b) + (b + c) = 2(a + b + c) = 2 \times 2011 = 4022.$$

35. Answer: 10.

By given, $BD + 2 + AE = BD + DC$. So $2 + AE = DC$.

Note that $AB^2 + BE^2 = AE^2$ and $BD^2 + BC^2 = DC^2$. Then

$$(2 + BD)^2 + 3^2 = AE^2, \quad BD^2 + 7^2 = (AE + 2)^2.$$

$4(AE + BD) = 32$. Then $AE + BA = \frac{32}{4} + 2 = 10$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Junior Section, Round 2)

Saturday, 25 June 2011 **0930-1230**

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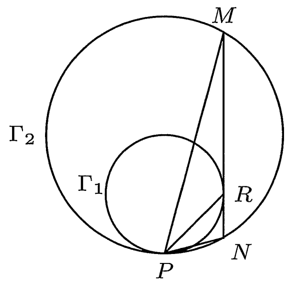
1. Suppose $a, b, c, d > 0$ and $x = \sqrt{a^2 + b^2}$, $y = \sqrt{c^2 + d^2}$. Prove that

$$xy \geq ac + bd.$$

2. Two circles Γ_1, Γ_2 with radii r_1, r_2, respectively, touch internally at the point P. A tangent parallel to the diameter through P touches Γ_1 at R and intersects Γ_2 at M and N. Prove that PR bisects $\angle MPN$.



3. Let $S_1, S_2, \ldots, S_{2011}$ be nonempty sets of consecutive integers such that any 2 of them have a common element. Prove that there is an integer that belongs to every S_i, $i = 1, \ldots, 2011$. (For example, $\{2, 3, 4, 5\}$ is a set of consecutive integers while $\{2, 3, 5\}$ is not.)

4. Any positive integer n can be written in the form $n = 2^a q$, where $a \geq 0$ and q is odd. We call q the *odd part* of n. Define the sequence $a_0, a_1, \ldots,$ as follows: $a_0 = 2^{2011} - 1$ and for $m \geq 0$, a_{m+1} is the odd part of $3a_m + 1$. Find a_{2011}.

5. Initially, the number 10 is written on the board. In each subsequent moves, you can either (i) erase the number 1 and replace it with a 10, or (ii) erase the number 10 and replace it with a 1 and a 25 or (iii) erase a 25 and replace it with two 10. After sometime, you notice that there are exactly one hundred copies of 1 on the board. What is the least possible sum of all the numbers on the board at that moment?

Singapore Mathematical Society

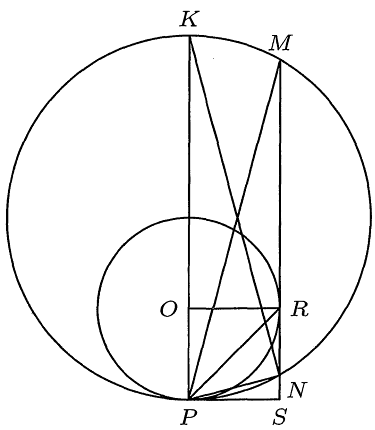
Singapore Mathematical Olympiad (SMO) 2011

(Junior Section, Round 2 solutions)

1. It's equivalent to prove $x^2y^2 \geq (ac+bd)^2$ as all the numbers are nonnegative. This is true since

$$\begin{aligned} x^2y^2 &= (a^2+b^2)(c^2+d^2) \\ &= (ac)^2+(bd)^2+a^2d^2+b^2c^2 \\ &\geq (ac)^2+(bd)^2+2(ac)(bd) \qquad \text{AM-GM} \\ &= (ac+bd)^2. \end{aligned}$$

2. Let the tangent at P meet the tangent at R at the point S. Let O be the centre of Γ_1. Then $ORST$ is a square. Hence $\angle KPR = \angle RPS = 45°$. Also $\angle NPS = \angle NKP = \angle PMS = \angle MPK$. Thus $\angle MPR = \angle RPN$.



3. Let $a_i = \max S_i$, $b_i = \min S_i$ and suppose that $t_1 = \min\{t_i\}$. For each j, if $S_1 \cap S_j \neq \emptyset$, then $a_1 \geq b_j$. Therefore $a_1 \in S_j$.

Note: Problem 4 in the Senior Section is the general version.

4. Replace 2011 by any positive odd integer n. We first show by induction that $a_m = 3^m 2^{n-m} - 1$ for $m = 0, 1, \ldots, n-1$. This is certainly true for $m = 0$. Suppose it's true for some $m < n-1$. Then $3a_m + 1 = 3^{m+1}2^{n-m} - 2$. Since $n - m > 1$, the odd part is $3^{m+1}2^{n-m-1} - 1$ which is a_{m+1}. Now $a_{n-1} = 3^{n-1}2^1 - 1$. Thus

$3a_{n-1}+1 = 3^n 2 - 2 = 2(3^n - 1)$. When n is odd, $3^n \equiv -1 \pmod 4$. Thus $4 \nmid 3^n - 1$. Hence the odd part of $2(3^n - 1)$ is $\frac{3^n-1}{2}$ and this is the value of a_n.

5. Suppose the number of times that operations (i), (ii) and (iii) have been performed are x, y and z, respectively. Then the number of 1, 10 and 25 are $y-x$, $1+x-y+2z$ and $y-z$, respectively, with $-x+y=100$. Thus the sum is

$$S = y - x + 10(1 + x - y + 2z) + 25(y - z) = -890 + 5(5y - z).$$

Since we want the minimum values of S, y has to be as small as possible and z as large as possible. Since

$$y - x = 100, \; 1 + x - y + 2z \geq 0, \quad y - z \geq 0$$

we get, from the first equation, $y \geq 100$, from the second inequality, $2z \geq 99$ or $z \geq 50$ and $y \geq z$ from the third. Thus the minimum is achieved when $y = 100$, $x = 0$ and $z = 100$. Thus minimum $S = 1100$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Senior Section)

Tuesday, 31 May 2011 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Suppose a, b and c are nonzero real numbers such that

$$\frac{a^2}{b^2+c^2} < \frac{b^2}{c^2+a^2} < \frac{c^2}{a^2+b^2}.$$

Which of the following statements is always true?

(A) $a<b<c$ (B) $|a|<|b|<|c|$ (C) $c<b<a$
(D) $|b|<|c|<|a|$ (E) $|c|<|b|<|a|$

2. Suppose θ is an angle between 0 and $\frac{\pi}{2}$, and $\sin 2\theta = a$. Which of the following expressions is equal to $\sin\theta + \cos\theta$?

(A) $\sqrt{a+1}$ (B) $(\sqrt{2}-1)a+1$ (C) $\sqrt{a+1}-\sqrt{a^2-a}$
(D) $\sqrt{a+1}+\sqrt{a^2-a}$ (E) None of the above

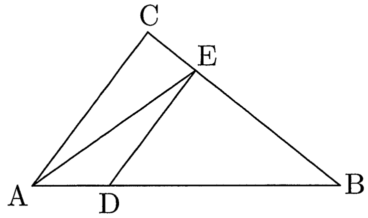
3. Let x be a real number. If $a = 2011x+9997$, $b = 2011x+9998$ and $c = 2011x+9999$, find the value of $a^2+b^2+c^2-ab-bc-ca$.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

4. Suppose x, y are real numbers such that $\frac{1}{x} - \frac{1}{2y} = \frac{1}{2x+y}$. Find the value of $\frac{y^2}{x^2} + \frac{x^2}{y^2}$.

(A) $\frac{2}{3}$ (B) $\frac{9}{2}$ (C) $\frac{9}{4}$ (D) $\frac{4}{9}$ (E) $\frac{2}{9}$

5. In the figure below, ABC is a triangle, and D and E are points on AB and BC respectively. It is given that DE is parallel to AC, and $CE : EB = 1 : 3$. If the area of $\triangle ABC$ is 1440 cm^2 and the area of $\triangle ADE$ is x cm^2, what is the value of x?



(A) 288 (B) 240 (C) 320 (D) 384 (E) 270

6. Determine the value of

$$\frac{2}{\dfrac{1}{\sqrt{2}+\sqrt[4]{8}+2}+\dfrac{1}{\sqrt{2}-\sqrt[4]{8}+2}}.$$

(A) $4-\sqrt{2}$ (B) $2-2\sqrt{2}$ (C) $4+\sqrt{2}$ (D) $2\sqrt{2}+4$
(E) $4\sqrt{2}-2$

7. Let $x=\dfrac{1}{\log_{\frac{1}{3}}\frac{1}{2}}+\dfrac{1}{\log_{\frac{1}{5}}\frac{1}{4}}+\dfrac{1}{\log_{\frac{1}{7}}\frac{1}{8}}$. Which of the following statements is true?

(A) $1.5 < x < 2$ (B) $2 < x < 2.5$ (C) $2.5 < x < 3$
(D) $3 < x < 3.5$ (E) $3.5 < x < 4$

8. Determine the last two digits of 7^{5^6}.

(A) 01 (B) 07 (C) 09 (D) 43 (E) 49

9. It is given that x and y are two real numbers such that $x > 1$ and $y > 1$. Find the smallest possible value of

$$\frac{\log_x 2011+\log_y 2011}{\log_{xy} 2011}.$$

(A) 4 (B) 6 (C) 8 (D) 10 (E) 12

10. It is given that a, b and c are three real numbers such that $a+b=c-1$ and $ab=c^2-7c+14$. Find the largest possible value of a^2+b^2.

(A) 5 (B) 6 (C) 8 (D) 9 (E) 10

Short Questions

11. Find the value of

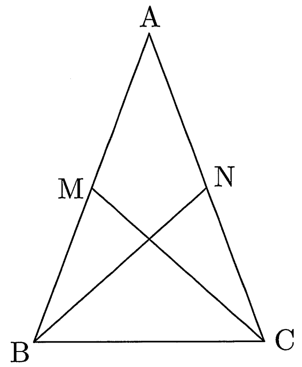
$$\frac{2011^2+2111^2-2\times 2011\times 2111}{25}.$$

12. Find the largest natural number n which satisfies the inequality

$$n^{6033} < 2011^{2011}.$$

13. Find the integer which is closest to $\dfrac{(1+\sqrt{3})^4}{4}$.

14. In the diagram below, $\triangle ABC$ is an isosceles triangle with $AB = AC$, and M and N are the midpoints of AB and AC respectively. It is given that CM is perpendicular to BN, $BC = 20$ cm, and the area of $\triangle ABC$ is x cm^2. Find the value of x.



15. Find the smallest positive integer n such that

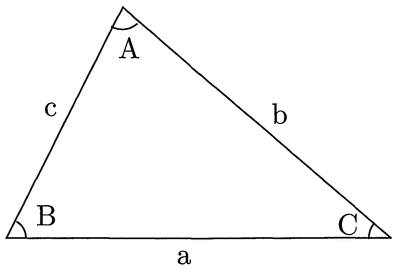
$$\sqrt{5n} - \sqrt{5n-4} < 0.01.$$

16. Find the value of

$$\frac{1}{1+11^{-2011}} + \frac{1}{1+11^{-2009}} + \frac{1}{1+11^{-2007}} + \cdots + \frac{1}{1+11^{2009}} + \frac{1}{1+11^{2011}}.$$

17. Let $x = \sin^4(\frac{\pi}{8}) + \cos^4(\frac{\pi}{8}) + \sin^4(\frac{7\pi}{8}) + \cos^4(\frac{7\pi}{8})$. Find the value of $36x$.

18. In the diagram below, the lengths of the three sides of the triangle are a cm, b cm and c cm. It is given that $\dfrac{a^2+b^2}{c^2} = 2011$. Find the value of $\dfrac{\cot C}{\cot A + \cot B}$.



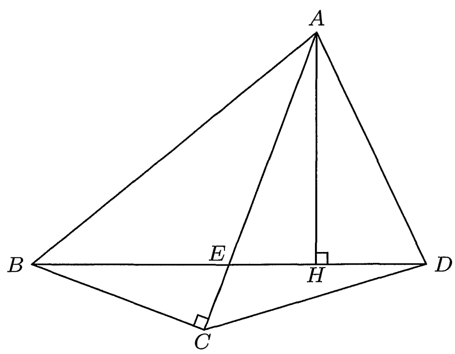
19. Suppose there are a total of 2011 participants in a mathematics competition, and at least 1000 of them are female. Moreover, given any 1011 participants, at least 11 of them are male. How many male participants are there in this competition?

20. Let $f : \mathbb{Q} \setminus \{0,1\} \to \mathbb{Q}$ be a function such that

$$x^2 f(x) + f\left(\frac{x-1}{x}\right) = 2x^2$$

for all rational numbers $x \neq 0, 1$. Here $\mathbb{Q}$ denotes the set of rational numbers. Find the value of $f(\frac{1}{2})$.

21. In the diagram below, $ABCD$ is a convex quadrilateral such that AC intersects BD at the midpoint E of BD. The point H is the foot of the perpendicular from A onto DE, and H lies in the interior of the segment DE. Suppose $\angle BCA = 90°$, $CE = 12$ cm, $EH = 15$ cm, $AH = 40$ cm and $CD = x$ cm. Find the value of x.



22. How many pairs of integers (x, y) are there such that

$$x \geq y \quad \text{and} \quad \frac{1}{x} + \frac{1}{y} = \frac{1}{211}?$$

23. It is given that a, b, c are three real numbers such that the roots of the equation $x^2+3x-1=0$ also satisfy the equation $x^4+ax^2+bx+c=0$. Find the value of $a+b+4c+100$.

24. It is given that m and n are two positive integers such that

$$n - \frac{m}{n} = \frac{2011}{3}.$$

Determine the smallest possible value of m.

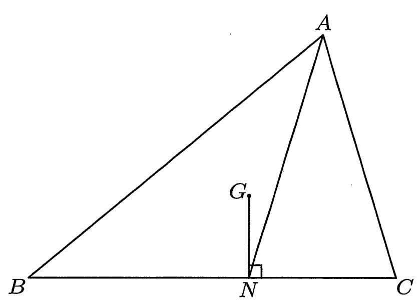
25. It is given that a, b, c are positive integers such that the roots of the three quadratic equations

$$x^2 - 2ax + b = 0, \quad x^2 - 2bx + c = 0, \quad x^2 - 2cx + a = 0$$

are all positive integers. Determine the maximum value of the product abc.

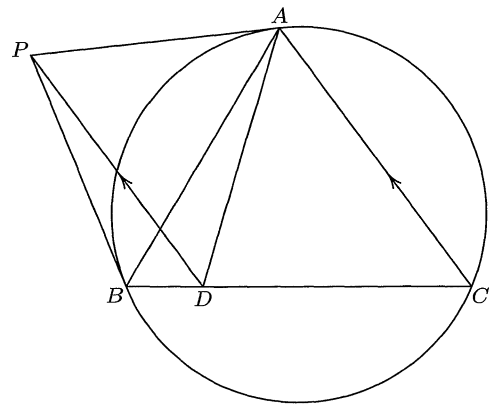
26. Suppose A, B, C are three angles such that $A \geq B \geq C \geq \frac{\pi}{8}$ and $A + B + C = \frac{\pi}{2}$. Find the largest possible value of the product $720 \times (\sin A) \times (\cos B) \times (\sin C)$.

27. In the diagram below, ABC is a triangle such that AB is longer than AC. The point N lies on BC such that AN bisects $\angle BAC$. The point G is the centroid of $\triangle ABC$, and it is given that GN is perpendicular to BC. Suppose $AC = 6$ cm, $BC = 5\sqrt{3}$ cm and $AB = x$ cm. Find the value of x.



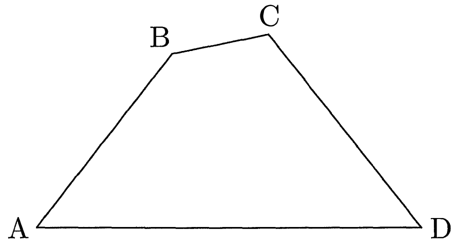
28. It is given that a, b, c and d are four positive prime numbers such that the product of these four prime numbers is equal to the sum of 55 consecutive positive integers. Find the smallest possible value of $a+b+c+d$. (Remark: The four numbers a, b, c, d are not necessarily distinct.)

29. In the diagram below, ABC is a triangle with $AB = 39$ cm , $BC = 45$ cm and $CA = 42$ cm. The tangents at A and B to the circumcircle of $\triangle ABC$ meet at the point P. The point D lies on BC such that PD is parallel to AC. It is given that the area of $\triangle ABD$ is x cm^2. Find the value of x.



30. It is given that a and b are positive integers such that a has exactly 9 positive divisors and b has exactly 10 positive divisors. If the least common multiple (LCM) of a and b is 4400, find the value of $a + b$.

31. In the diagram below, $ABCD$ is a quadrilateral such that $\angle ABC = 135°$ and $\angle BCD = 120°$. Moreover, it is given that $AB = 2\sqrt{3}$ cm, $BC = 4 - 2\sqrt{2}$ cm, $CD = 4\sqrt{2}$ cm and $AD = x$ cm. Find the value of $x^2 - 4x$.



32. It is given that p is a prime number such that

$$x^3 + y^3 - 3xy = p - 1$$

for some positive integers x and y. Determine the largest possible value of p.

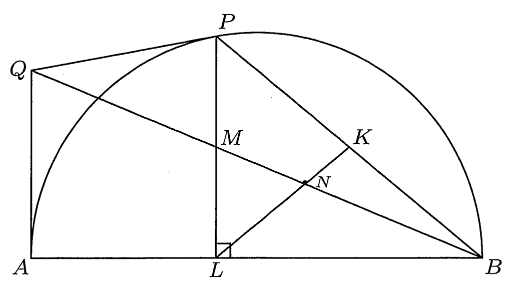
33. It is given that a, b and c are three positive integers such that

$$a^2 + b^2 + c^2 = 2011.$$

Let the highest common factor (HCF) and the least common multiple (LCM) of the three numbers a, b, c be denoted by x and y respectively. Suppose that $x + y = 388$. Find the value of $a + b + c$. (Remark: The highest common factor is also known as the greatest common divisor.)

34. Consider the set $S = \{1, 2, 3, \cdots, 2010, 2011\}$. A subset T of S is said to be a k-element RP-subset if T has exactly k elements and every pair of elements of T are relatively prime. Find the smallest positive integer k such that every k-element RP-subset of S contains at least one prime number. (As an example, $\{1, 8, 9\}$ is a 3-element RP-subset of S which does not contain any prime number.)

35. In the diagram below, P is a point on the semi-circle with diameter AB. The point L is the foot of the perpendicular from P onto AB, and K is the midpoint of PB. The tangents to the semicircle at A and at P meet at the point Q. It is given that PL intersects QB at the point M, and KL intersects QB at the point N. Suppose $\dfrac{AQ}{AB} = \dfrac{5}{12}$, $QM = 25$ cm and $MN = x$ cm. Find the value of x.



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(Senior Section Solutions)

1. Answer: (B)

   Since a, b and c are nonzero real numbers and $\dfrac{a^2}{b^2+c^2} < \dfrac{b^2}{c^2+a^2} < \dfrac{c^2}{a^2+b^2}$, we see that
$$\frac{b^2+c^2}{a^2} > \frac{c^2+a^2}{b^2} > \frac{a^2+b^2}{c^2}.$$
   Adding 1 throughout, we obtain
$$\frac{a^2+b^2+c^2}{a^2} > \frac{a^2+b^2+c^2}{b^2} > \frac{a^2+b^2+c^2}{c^2}.$$
   Thus $\dfrac{1}{a^2} > \dfrac{1}{b^2} > \dfrac{1}{c^2}$, which implies that $a^2 < b^2 < c^2$. So we have $|a| < |b| < |c|$.

2. Answer: (A)

   Note that
$$(\sin\theta + \cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1 + \sin 2\theta = 1 + a.$$
   Since $0 \leq \theta \leq \frac{\pi}{2}$, we have $\sin\theta + \cos\theta > 0$. So $\sin\theta + \cos\theta = \sqrt{1+a}$.

3. Answer: (D)

   We have
$$\begin{aligned} a^2+b^2+c^2-ab-bc-ca &= \frac{1}{2}\cdot[(a-b)^2+(b-e)^2+(c-a)^2] \\ &= \frac{1}{2}\cdot[(-1)^2+(-1)^2+2^2] = 3. \end{aligned}$$

4. Answer: (C)
$$\begin{aligned} \frac{1}{x} - \frac{1}{2y} = \frac{1}{2x+y} \quad &\Rightarrow \quad \frac{2x+y}{x} - \frac{2x+y}{2y} = 1 \\ &\Rightarrow \quad 2 + \frac{y}{x} - \frac{x}{y} - \frac{1}{2} = 1 \\ &\Rightarrow \quad \frac{y}{x} - \frac{x}{y} = -\frac{1}{2}. \end{aligned}$$
   Now we have
$$\frac{y^2}{x^2} + \frac{x^2}{y^2} = \left(\frac{y}{x} - \frac{x}{y}\right)^2 + 2 = \left(-\frac{1}{2}\right)^2 + 2 = \frac{9}{4}.$$

5. Answer: (E)

The area of $\triangle ABC$ is given to be $S = 1440\text{ cm}^2$. Let S_1 and S_2 denote the areas of $\triangle ADE$ and $\triangle DBE$ respectively. Since DE is parallel to AC, $\triangle DBE$ and $\triangle ABC$ are similar. Therefore

$$\frac{S_2}{S} = \left(\frac{BE}{BC}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}.$$

Thus $S_2 = \frac{9}{16}S$. As DE is parallel to AC, we have $AD : DB = CE : EB = 1 : 3$. Consequently,

$$\frac{S_1}{S_2} = \frac{1}{3}.$$

Hence $S_1 = \dfrac{1}{3}S_2 = \dfrac{1}{3}\cdot\dfrac{9}{16}S = \dfrac{3}{16}\times 1440 = 270\text{ cm}^2.$

6. Answer: (A)

Let $x = 2^{\frac{1}{4}}$. Then

$$\begin{aligned}
\frac{2}{\frac{1}{2^{\frac{1}{2}}+2^{\frac{3}{4}}+2}+\frac{1}{2^{\frac{1}{2}}-2^{\frac{3}{4}}+2}} &= \frac{2}{\frac{1}{x^2+x^3+2}+\frac{1}{x^2-x^3+2}} \\
&= \frac{2}{\frac{2(x^2+2)}{(x^2+2)^2-x^6}} \\
&= \frac{(x^2+2)^2-x^6}{x^2+2} \\
&= \frac{(\sqrt{2}+2)^2-2\sqrt{2}}{\sqrt{2}+2} \\
&= \frac{6+2\sqrt{2}}{\sqrt{2}+2}\times\frac{2-\sqrt{2}}{2-\sqrt{2}} \\
&= \frac{8-2\sqrt{2}}{2} \\
&= 4-\sqrt{2}.
\end{aligned}$$

7. Answer: (E)

$$\begin{aligned}
x &= \frac{\log(\frac{1}{3})}{\log(\frac{1}{2})}+\frac{\log(\frac{1}{5})}{\log(\frac{1}{4})}+\frac{\log(\frac{1}{7})}{\log(\frac{1}{8})} = \frac{-\log 3}{-\log 2}+\frac{-\log 5}{-\log 4}+\frac{-\log 7}{-\log 8} \\
&= \frac{\log 3+\log 5^{\frac{1}{2}}+\log 7^{\frac{1}{3}}}{\log 2} \\
&= \frac{\log\sqrt{45}+\log 7^{\frac{1}{3}}}{\log 2} < \frac{\log\sqrt{64}+\log 8^{\frac{1}{3}}}{\log 2} = \frac{3\log 2+\log 2}{\log 2} = 4.
\end{aligned}$$

Moreover,

$$\begin{aligned}
2x &= \frac{2(\log 3 + \log 5^{\frac{1}{2}} + \log 7^{\frac{1}{3}})}{\log 2} \\
&= \frac{\log(9 \times 5) + \log(49^{\frac{1}{3}})}{\log 2} > \frac{\log(45 \times 27^{\frac{1}{3}})}{\log 2} \\
&= \frac{\log(45 \times 3)}{\log 2} > \frac{\log(128)}{\log 2} = 7,
\end{aligned}$$

so $x > 3.5$.

8. Answer: (B)

   Note that $7^4 - 1 = 2400$, so that $7^{4n} - 1$ is divisible by 100 for any $n \in \mathbb{Z}^+$. Now,

$$\begin{aligned}
7^{5^6} &= 7(7^{5^6-1} - 1 + 1) \\
&= 7(7^{5^6-1} - 1) + 7 \\
&= 7(7^{4n} - 1) + 7,
\end{aligned}$$

   where

$$n = \frac{5^6 - 1}{4} \in \mathbb{Z}^+.$$

   Since $7(7^{4n} - 1)$ is divisible by 100, its last two digits are 00. It follows that the last two digits of 7^{5^6} are 07.

9. Answer: (A)

$$\begin{aligned}
\frac{\log_x 2011 + \log_y 2011}{\log_{xy} 2011} &= (\frac{\log 2011}{\log x} + \frac{\log 2011}{\log y}) \cdot (\frac{\log xy}{\log 2011}) \\
&= (\frac{1}{\log x} + \frac{1}{\log y}) \cdot (\log x + \log y) \\
&= 2 + \frac{\log x}{\log y} + \frac{\log y}{\log x} \\
&\geq 4 \text{ (using } AM \geq GM),
\end{aligned}$$

   and the equality is attained when $\log x = \log y$, or equivalently, $x = y$.

10. Answer: (C)

    The roots of the equation $x^2 - (c-1)x + c^2 - 7c + 14 = 0$ are a and b, which are real. Thus the discriminant of the equation is non-negative. In other words,

$$(c-1)^2 - 4(c^2 - 7c + 14) = -3c^2 + 26c - 55 = (-3c + 11)(c - 5) \geq 0.$$

    So we have $\frac{11}{3} \leq c \leq 5$. Together with the equalities

$$\begin{aligned}
a^2 + b^2 &= (a+b)^2 - 2ab \\
&= (c-1)^2 - 2(c^2 - 7c + 14) \\
&= -c^2 + 12c - 27 = 9 - (c-6)^2,
\end{aligned}$$

we see that maximum value of $a^2 + b^2$ is 8 when $c = 5$.

11. Answer: 400

$$\frac{2011^2 + 2111^2 - 2 \times 2011 \times 2111}{25} = \frac{(2011 - 2111)^2}{25} = \frac{100^2}{25} = 400.$$

12. Answer: 12

Since $6033 = 3 \times 2011$, we have

$$\begin{aligned} n^{6033} < 2011^{2011} &\iff n^{3\times 2011} < 2011^{2011} \\ &\iff n^3 < 2011. \end{aligned}$$

Note that $12^3 = 1728$ and $13^3 = 2197 > 2011$. Thus, the largest possible natural number n satisfying the given inequality is 12.

13. Answer: 14

$$(1 + \sqrt{3})^2 = 1^2 + 3 + 2\sqrt{3} = 4 + 2\sqrt{3}.$$

$$\begin{aligned} (1 + \sqrt{3})^4 = ((1 + \sqrt{3})^2)^2 = (4 + 2\sqrt{3})^2 &= 4^2 + (2\sqrt{3})^2 + 2 \cdot 4 \cdot 2\sqrt{3} \\ &= 16 + 12 + 16\sqrt{3} \\ &= 28 + 16\sqrt{3}. \end{aligned}$$

Hence,
$$\frac{(1 + \sqrt{3})^4}{4} = \frac{28 + 16\sqrt{3}}{4} = 7 + 4\sqrt{3}.$$

Note that $1.7 < \sqrt{3} < 1.8$. Thus,

$$\begin{aligned} & 7 + 4 \times 1.7 < 7 + 4\sqrt{3} < 7 + 4 \times 1.8 \\ \Longrightarrow \quad & 13.8 < 7 + 4\sqrt{3} < 14.2. \end{aligned}$$

Therefore, the integer closest to $\frac{(1+\sqrt{3})^4}{4}$ is 14.

14. Answer: 300

Let P be the intersection of CM and BN, so that P is the centroid of $\triangle ABC$. Then $BP = 2PN$ and $CP = 2PM$. Let $PN = PM = y$, so that $BP = CP = 2y$. Since $\angle BPC = 90°$, we have $BC = 20 = \sqrt{(2y)^2 + (2y)^2}$ and thus $y = \sqrt{50}$. Now

$$AB^2 = 4BM^2 = 4((2y)^2 + y^2) = 20y^2 = 1000.$$

Thus the altitude of $\triangle ABC$ is $\sqrt{AB^2 - 10^2} = 30$. Hence the area of $\triangle ABC$ is $\frac{1}{2} \times 20 \times 30 = 300$.

15. Answer: 8001

Note that $\sqrt{5n} - \sqrt{5n-4} < 0.01$ if and only if

$$\sqrt{5n} + \sqrt{5n-4} = \frac{4}{\sqrt{5n} - \sqrt{5n-4}} > 400.$$

If $n = 8000$, then $\sqrt{5n} + \sqrt{5n-4} = \sqrt{40000} + \sqrt{39996} < 400$.

If $n = 8001$, then $\sqrt{5n} + \sqrt{5n-4} = \sqrt{40005} + \sqrt{40001} > 400$.

So the answer is 8001.

16. Answer: 1006

The series can be paired as

$$(\frac{1}{1+11^{-2011}} + \frac{1}{1+11^{2011}}) + (\frac{1}{1+11^{-2009}} + \frac{1}{1+11^{2009}}) + \cdots + (\frac{1}{1+11^{-1}} + \frac{1}{1+11^{1}}).$$

Each pair of terms is of the form

$$\frac{1}{1+a^{-1}} + \frac{1}{1+a} = 1.$$

There are 1006 pairs of such terms, and thus the sum of the series is 1006.

17. Answer: 54

$$\begin{aligned} x &= \sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{7\pi}{8}\right) + \cos^4\left(\frac{7\pi}{8}\right) \\ &= \sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) \\ &= 2(\sin^2\left(\frac{\pi}{8}\right) + \cos^2\left(\frac{\pi}{8}\right))^2 - 4\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right) = 2 - \sin^2\left(\frac{\pi}{4}\right) = \frac{3}{2}. \end{aligned}$$

Thus $36x = 54$.

18. Answer: 1005

By the laws of sine and cosine, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{and} \quad \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

Then

$$\begin{aligned} \frac{\cot C}{\cot A + \cot B} &= \frac{\cos C}{\sin C} \cdot \frac{1}{\frac{\cos A \sin B + \cos B \sin A}{\sin A \sin B}} \\ &= \frac{\sin A \sin B \cos C}{\sin(A+B)\sin C} \\ &= (\frac{\sin A \sin B}{\sin^2 C})\cos C \\ &= (\frac{(ab/c^2)\sin^2 C}{\sin^2 C})(\frac{a^2+b^2-c^2}{2ab}) \\ &= \frac{a^2+b^2-c^2}{2c^2} \\ &= \frac{2011-1}{2} = 1005. \end{aligned}$$

19. Answer: 1011

It is given that there are at least 1000 female participants. Suppose there are more than 1000 female participants. Then we take a group of 1001 female participants, and add any 10 participants to this group of female participants. This will result in a group of 1011 participants with at most 10 male participants, which contradicts the assumption. Therefore, there are exactly 1000 female participants. Hence, the number of male participants is $2011 - 1000 = 1011$.

20. Answer: 1

Substituting $x = \frac{1}{2}, -1, 2$, we get

$$\begin{array}{ccccccc} \frac{1}{4}f(\frac{1}{2}) & + & f(-1) & & & = & \frac{1}{2}, \\ & & f(-1) & + & f(2) & = & 2, \\ f(\frac{1}{2}) & + & & & 4f(2) & = & 8. \end{array}$$

Solving these equations, we get $f(\frac{1}{2}) = 1$. In fact the same method can be used to determine f. Letting $x = z, \frac{z-1}{z}, \frac{1}{1-z}$, we get

$$\begin{array}{ccccccc} z^2 f(z) & + & f\left(\frac{z-1}{z}\right) & & & = & 2z^2, \\ & & \left(\frac{z-1}{z}\right)^2 f\left(\frac{z-1}{z}\right) & + & f\left(\frac{1}{1-z}\right) & = & 2\left(\frac{z-1}{z}\right)^2, \\ f(z) & + & & & \frac{1}{(1-z)^2} f\left(\frac{1}{1-z}\right) & = & \frac{1}{2(1-z)^2}. \end{array}$$

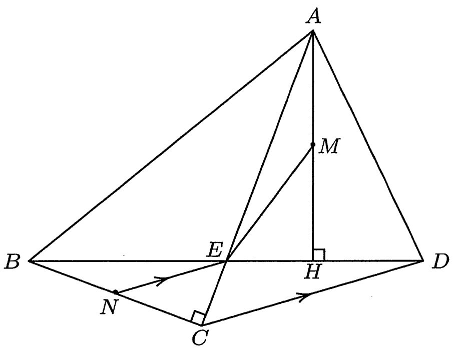
Using Cramer's rule, we can solve this system of linear equations in the unknowns $f(z), f\left(\frac{z-1}{z}\right), f\left(\frac{1}{1-z}\right)$. We obtain

$$f(z) = 1 + \frac{1}{(1-z)^2} - \frac{1}{z^2}.$$

Indeed one can easily check that it satisfies the given functional equation.

21. Answer: 40

Let M be the midpoint of AH and N the midpoint of BC. Then CD is parallel to NE and $CD = 2NE$. Since $\triangle AHE$ is similar to $\triangle BCE$, we have $\triangle MHE$ is similar to $\triangle NCE$. As $ME = \sqrt{15^2 + 20^2} = 25$, we thus have $NE = \frac{EC}{EH} \cdot ME = \frac{12\times 25}{15} = 20$. Therefore $CD = 2NE = 40$.



22. Answer: 3

Note that

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{211}\Rightarrow xy-211x-211y=0\Rightarrow (x-211)(y-211)=211^2.$$

Since 211 is a prime number, the factors of 211^2 are 1, 211, 211^2, -1, -211, -211^2. Thus the pairs of integers (x, y) satisfying the last equation are given by:

$$(x-211,\ y-211)=(1,\ 211^2),\ (211,\ 211),\ (211^2,\ 1),\ (-1,\ -211^2),$$
$$(-211,\ -211),\ (-211^2,\ -1).$$

Equivalently, (x, y) are given by

$$(212,\ 211+211^2),\ (422,\ 422),\ (211+211^2,\ 212),\ (210,\ 211-211^2),$$
$$(0,\ 0),\ (211-211^2,\ 210).$$

Note that $(0,\ 0)$ does not satisfy the first equation. Among the remaining 5 pairs which satisfy the first equation, three of them satisfy the inequality $x \geq y$, and they are given by $(x,\ y)=(422,\ 422),\ (211+211^2,\ 212),\ (210,\ 211-211^2)$.

23. Answer: 93

By long division, we have

$$x^4+ax^2+bx+c=(x^2+3x-1)\cdot(x^2-3x+(a+10))+(b-3a-33)x+(c+a+10).$$

Let m_1, m_2 be the two roots of the equation $x^2+3x-1=0$. Note that $m_1 \neq m_2$, since the discriminant of the above quadratic equation is $3^2-4\cdot1\cdot1\cdot(-1)=13\neq 0$. Since m_1, m_2 also satisfy the equation $x^4+ax^2+bx+c=0$, it follows that m_1 and m_2 also satisfy the equation

$$(b-3a-33)x+(c+a+10)=0.$$

Thus we have

$$(b-3a-33)m_1+(c+a+10)=0,$$

and

$$(b-3a-33)m_2+(c+a+10)=0.$$

Since $m_1 \neq m_2$, it follows that $b-3a-33=0$ and $c+a+10=0$. Hence we have $b=3a+33$ and $c=-a-10$. Thus $a+b+4c+100=a+(3a+33)+4(-a-10)+100=93$.

24. Answer: 1120

Let m and n be positive integers satisfying the given equation. Then $3(n^2-m)=2011n$. Since 2011 is a prime, 3 divides n. By letting $n=3k$, we have $(3k)^2=m+2011k$. This implies that k divides m. Let $m=rk$. Then $9k^2=rk+2011k$ so that $9k=r+2011$. The smallest positive integer r such that $r+2011$ is divisible by 9 is $r=5$. Thus $k=(5+2011)/9=224$. The corresponding values of m and n are $m=1120$ and $n=672$.

25. Answer: 1

We shall show that the only possible values of a, b, c are $a = b = c = 1$ so that $abc = 1$. From the first equation, we note that $a^2 - b = (x - a)^2$ is a perfect square less than a^2. Thus $a^2 - b \leq (a-1)^2$. That is $b \geq 2a - 1$. Likewise $c \geq 2b - 1$ and $a \geq 2c - 1$. Combining these inequalities, we have $a \geq 8a - 7$ or $a \leq 1$. Thus $a = 1$. Similarly $b = c = 1$.

26. Answer: 180

Note that

$$A = \frac{\pi}{2} - B - C \leq \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$

and $\sin(B - C) \geq 0$. Therefore

$$\begin{aligned} \sin A \cos B \sin C &= \frac{1}{2} \cdot \sin A \cdot [\sin(B + C) - \sin(B - C)] \\ &\leq \frac{1}{2} \sin A \sin(B + C) \\ &= \frac{1}{2} \sin A \cos A \\ &= \frac{1}{4} \sin 2A \leq \frac{1}{4} \sin \frac{\pi}{2} = \frac{1}{4}. \end{aligned}$$

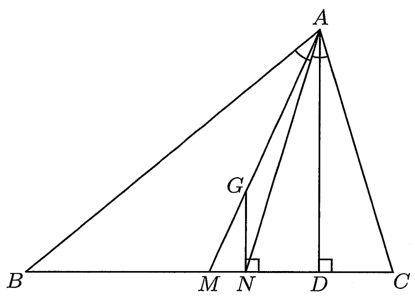
When $A = \frac{\pi}{4}$ and $B = C = \frac{\pi}{8}$, we have

$$\sin A \cos B \sin C = \sin \frac{\pi}{4} \cos \frac{\pi}{8} \sin \frac{\pi}{8} = \sin \frac{\pi}{4} \cdot \frac{1}{2} \sin \frac{\pi}{4} = \frac{1}{4}.$$

Hence the largest possible value of $720(\sin A)(\cos B)(\sin C)$ is $720 \times \frac{1}{4} = 180$.

27. Answer: 9

Let $BC = a, CA = b$ and $AB = c$. We shall prove that $c + b = a\sqrt{3}$. Thus $c = 5\sqrt{3} \times \sqrt{3} - 6 = 9$. Using the angle bisector theorem, we have $BN/NC = c/a > 1$ so that $BN > BM$. Also $\angle ANC = \angle B + \frac{1}{2}\angle A < \angle C + \frac{1}{2}\angle A < \angle ANB$ so that $\angle ANC$ is acute. This shows that B, M, N, D are in this order, where M is the midpoint of BC and D is the foot of the perpendicular from A onto BC.



First we have $BD = c \cos B = \frac{a^2+c^2-b^2}{2a}$. Using the angle bisector theorem, $BN = \frac{ac}{b+c}$. Therefore, $MN = BN - \frac{a}{2} = \frac{ac}{b+c} - \frac{a}{2} = \frac{a(c-b)}{2(c+b)}$. Also $MD = BD - \frac{a}{2} = \frac{a^2+c^2-b^2}{2a} -$

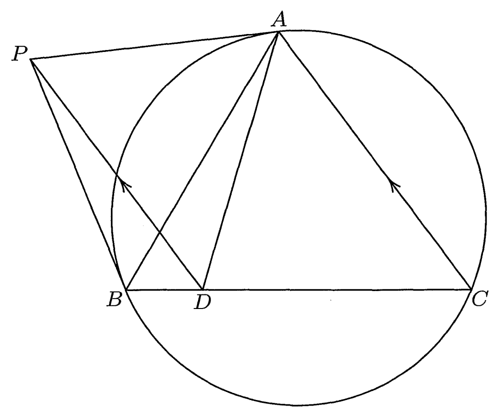
$\frac{a}{2} = \frac{c^2-b^2}{2a}$. Since GN is parallel to AD and G is the centroid of the triangle ABC, we have $MD/MN = 3$. It follows that $c+b = a\sqrt{3}$. Thus, $AB = a\sqrt{3} - b = 15 - 6 = 9$.

28. Answer: 28

The sum of 55 positive consecutive integers is at least $(55 \times 56)/2 = 1540$. Let the middle number of these consecutive positive integers be x. Then the product $abcd = 55x = 5 \cdot 11 \cdot x$. So we have $55x \geq 1540$ and thus $x \geq 28$. The least value of $a + b + c + d$ is attained when $x = 5(7)$. Thus the answer is $5 + 11 + 5 + 7 = 28$.

29. Answer: 168

First $\angle BDP = \angle BCA = \angle BAP$ so that P, B, D, A are concyclic. Thus $\angle ACD = \angle PBA = \angle PDA = \angle DAC$ so that $DA = DC$.



By cosine rule, $\cos C = 3/5$. Thus $DC = \frac{1}{2}AC/\cos C = 21 \times 5/3 = 35$. Hence $BD = 10$ and $BC = 10 + 35 = 45$. Thus $\text{area}(\triangle ABD) = \frac{10}{45} \times \text{area}(\triangle ABC)$. By Heron's formula, $\text{area}(\triangle ABC) = 756$. Thus $\text{area}(\triangle ABD) = \frac{10}{45} \times 756 = 168$.

30. Answer: 276

Since the number of positive divisors of a is odd, a must be a perfect square. As a is a divisor of $4400 = 2^4 \times 5^2 \times 11$ and a has exactly 9 positive divisors, we see that $a = 2^2 \times 5^2$. Now the least common multiple of a and b is 4400 implies that b must have $2^4 \times 11$ as a divisor. Since $2^4 \times 11$ has exactly 10 positive divisors, we deduce that $b = 2^4 \times 11 = 176$. Hence $a + b = 276$.

31. Answer: 20

First we let ℓ be the line which extends BC in both directions. Let E be the point on ℓ such that AE is perpendicular to ℓ. Similarly, we let F be the point on ℓ such that DF is perpendicular to ℓ. Then, it is easy to see that $BE = AE = \sqrt{6}$, $CF = 2\sqrt{2}$ and $DF = 2\sqrt{6}$. Thus $EF = \sqrt{6} + 4 - 2\sqrt{2} + 2\sqrt{2} = 4 + \sqrt{6}$. Now we let G be the point on DF such that AG is parallel to ℓ. Then $AG = EF = 4 + \sqrt{6}$ and

$DG = DF - GF = DF - AE = 2\sqrt{6} - \sqrt{6} = \sqrt{6}$. So for the right-angled triangle ADG, we have

$$x = AD = \sqrt{AG^2 + DG^2} = \sqrt{(4+\sqrt{6})^2 + (\sqrt{6})^2} = \sqrt{28 + 8\sqrt{6}} = 2 + 2\sqrt{6}.$$

Thus, $x^2 - 4x = (2+2\sqrt{6})^2 - 8 - 8\sqrt{6} = 4 + 24 + 8\sqrt{6} - 8 - 8\sqrt{6} = 20$.

32. Answer: 5

Suppose x and y are positive integers satisfying $x^3 + y^3 - 3xy = p - 1$. That is $(x+y+1)(x^2+y^2-xy-x-y+1) = p$. Since $x, y \geq 1$, we must have $x+y+1 = p$ and $x^2+y^2-xy-x-y+1 = 1$. Suppose $x = y$. Then the second equation gives $x = y = 2$. Thus $p = x+y+1 = 5$. Next we may suppose without loss of generality that $x > y \geq 1$. Thus $x - y \geq 1$. Then the equation $x^2+y^2-xy-x-y+1 = 1$ can be written as $x + y - xy = (x-y)^2 \geq 1$. That is $(x-1)(y-1) \leq 0$. It follows that $x = 1$ or $y = 1$. Since we assume $x > y$, we must have $y = 1$. Then from $x^2+y^2-xy-x-y+1 = 1$, we get $x = 2$. But then $x+y+1 = 4$ is not a prime. Consequently there is no solution in x and y if $x \neq y$. Therefore the only solution is $x = y = 2$ and $p = 5$.

33. Answer: 61

Without loss of generality, we may assume that $a \geq b \geq c$. Let the HCF (or GCD) of a, b and c be d. Then $a = da_1$, $b = db_1$ and $c = dc_1$. Let the LCM of a_1, b_1 and c_1 be m. Thus, $a_1^2 + b_1^2 + c_1^2 = \frac{2011}{d^2}$ and $d + md = 388$ or $1 + m = \frac{388}{d}$. So, $d^2 \mid 2011$ and $d \mid 388$. Note that 2011 is a prime. Thus we must have $d = 1$, and it follows that $a = a_1$, $b = b_1$, $c = c_1$, and thus $a^2 + b^2 + c^2 = 2011$. In particular, $a^2 + b^2 + c^2 < 2025 = 45^2$, so that one has $a, b, c < 45$. Furthermore we have $m = 387 = 3^2 \times 43$. Thus a, b and c can only be 1, 3, 9 or 43, since they must be less than 45. Then it is easy to check that $43^2 + 9^2 + 9^2 = 2011$, and $a = 43$, $b = c = 9$ is the only combination which satisfies the given conditions. Thus we have $a + b + c = 43 + 18 = 61$.

34. Answer: 16

Consider the subset $T = \{1, 2^2, 3^2, 5^2, 7^2, \cdots, 43^2\}$ consisting of the number 1 and the squares of all prime numbers up to 43. Then $T \subseteq S$, $|T| = 15$, and all elements in T are pairwise relatively prime; however, T contains no prime number. Thus $k \geq 16$. Next we show that if A is any subset of S with $|A| = 16$ such that all elements in A are pairwise relatively prime, then A contains a prime number. Suppose to the contrary that A does not contain a prime number. Let $A = \{a_1, a_2, \cdots, a_{16}\}$. We consider two cases:

Case 1. 1 is not in A. Then $a_1, a_2, \cdots, a_{16}$ are composite numbers. Let p_i be the smallest prime factor of a_i, $i = 1, 2, \cdots, 16$. Since $\gcd(a_i, a_j) = 1$ for $i \neq j$, $i, j = 1, 2, \cdots, 16$, we see that the prime numbers $p_1, p_2, \cdots, p_{16}$ are all distinct. By re-ordering $a_1, a_2, \cdots, a_{16}$ if necessary, we may assume that $p_1 < p_2 < \cdots < p_{16}$. In addition, since each a_i is a composite number, it follows that

$$a_1 \geq p_1^2 \geq 2^2,\ a_2 \geq p_2^2 \geq 3^2, \cdots,\ a_{15} \geq p_{15}^2 \geq 47^2 > 2011,$$

which is a contradiction to the given fact that each element of S is less than or equal to 2011.

Case 2. 1 is in A. We may let $a_{16} = 1$. Then $a_1, a_2, \cdots, a_{15}$ are composite numbers. As in Case 1, we have

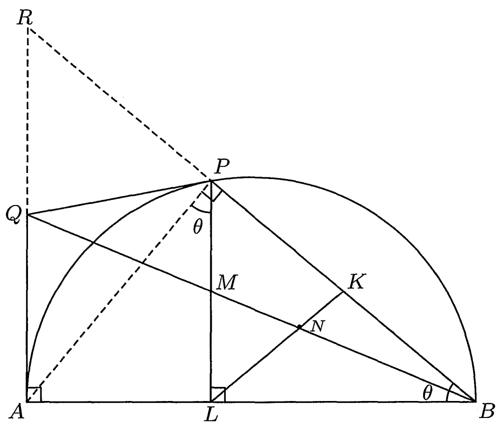
$$a_1 \geq p_1^2 \geq 2^2,\ a_2 \geq p_2^2 \geq 3^2, \cdots,\ a_{15} \geq p_{15}^2 \geq 47^2 > 2011,$$

which is a contradiction.

Thus we have shown that every 16-element subset A of S such that all elements in A are pairwise relatively prime must contain a prime number. Hence the smallest k is 16.

35. Answer: 12

Let the extensions of AQ and BP meet at the point R. Then $\angle PRQ = \angle PAB = \angle QPR$ so that $QP = QR$. Since $QA = QP$, the point Q is the midpoint of AR. As AR is parallel to LP, the triangles ARB and LPB are similar so that M is the midpoint of PL. Therefore, N is the centroid of the triangle PLB, and $3MN = BM$.



Let $\angle ABP = \theta$. Thus $\tan\theta = AR/AB = 2AQ/AB = 5/6$. Then $BL = PB\cos\theta = AB\cos^2\theta$. Also $BM/BL = BQ/BA$ so that $3MN = BM = \frac{BQ}{AB}AB\cos^2\theta = \cos^2\theta(QM + 3MN)$. Solving for MN, we have $MN = \frac{QM}{3\tan^2\theta} = \frac{25}{3\times(5/6)^2} = 12$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Senior Section, Round 2)

Saturday, 25 June 2011 **0930-1230**

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1. In the triangle ABC, the altitude at A, the bisector of $\angle B$ and the median at C meet at a common point. Prove that the triangle ABC is equilateral.

2. Determine if there is a set S of 2011 positive integers so that for every pair m, n of distinct elements of S, $|m - n| = (m, n)$. Here (m, n) denotes the greatest common divisor of m and n.

3. Find all positive integers n such that

$$\cos\frac{\pi}{n}\cos\frac{2\pi}{n}\cos\frac{3\pi}{n} = \frac{1}{n+1}.$$

4. Let n and k be positive integers with $n \geq k \geq 2$. For $i = 1, \ldots, n$, let S_i be a nonempty set of consecutive integers such that among any k of them, there are two with nonempty intersection. Prove that there is a set X of $k-1$ integers such that each S_i, $i = 1, \ldots, n$ contains at least one integer in X.

5. Given $x_1, x_2, \cdots, x_n > 0$, $n \geq 5$, show that

$$\frac{x_1x_2}{x_1^2+x_2^2+2x_3x_4} + \frac{x_2x_3}{x_2^2+x_3^2+2x_4x_5} + \cdots + \frac{x_nx_1}{x_n^2+x_1^2+2x_2x_3} \leq \frac{n-1}{2}.$$

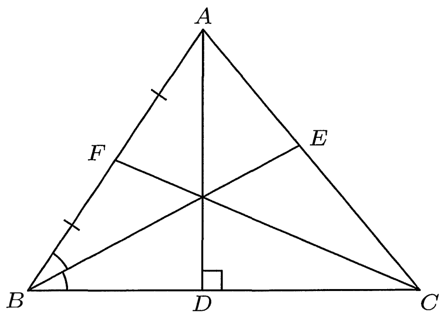
Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Senior Section, Round 2 solutions)

1. There is an error in this problem. The triangle is not necessarily equilateral. In fact we shall prove that the altitude at A, the bisector of $\angle B$ and the median at C meet at a common point if and only if $\cos B = \frac{a}{a+c}$ where $BC = a$, $CA = b$ and $AB = c$.

Let D, E and F be the points on BC, CA and AB respectively such that AD is the altitude at A, BE is the bisector of $\angle B$ and CF is the median at C. Suppose that AD, BE, CF meet at a common point. The point of concurrence of AD, BE and CF must lie inside the triangle ABC. Since F is the midpoint of AB, by Ceva's theorem $CE : EA = CD : DB$. Using the angle bisector theorem, $CE : EA = a : c$. Thus $CD = a^2/(a+c)$ and $DB = ac/(a+c)$. Thus $\cos B = \frac{BD}{AB} = \frac{a}{a+c}$.



Conversely, if $\cos B = \frac{a}{a+c}$, then $\angle B$ is acute and $BD = c\cos B = ac/(a+c) < a$ so that D is within BC. Thus $DC = a - ac/(a+c) = a^2/(a+c)$. Therefore $BD/DC = c/a$. Consequently $(AF/FB)(BD/DC)(CE/EA) = 1$. By Ceva's theorem, AD, BE and CF are concurrent.

So given a and c, the acute angle B and hence the triangle ABC is determined. If $a \neq c$, then the triangle ABC is not equilateral.

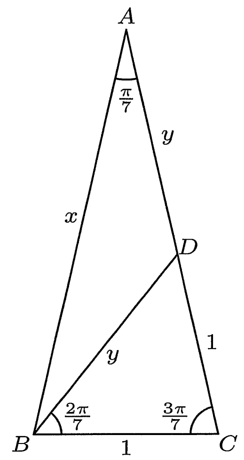
2. Yes, in fact, for any $k \in \mathbb{N}$, there is a set S_k having k elements satisfying the given condition. For $k = 1$, let S_1 be any singleton set. For $k = 2$, let $S_2 = \{2, 3\}$. Suppose that $S_k = \{a_1, \ldots, a_k\}$ satisfies the given conditions. Let

$$b_1 = a_1 a_2 \cdots a_k$$
$$b_j = b_1 + a_{j-1}, \quad 2 \leq j \leq k+1.$$

Let $S_{k+1} = \{b_1, b_2, \ldots, b_{k+1}\}$. Then the fact that S_{k+1} satisfies the required property can be verified by observing that $|m - n| = (m, n)$ if and only if $(m - n)$ divides m.

3. We shall show that $n = 3$ or 7. Let $f(n) = \cos\frac{\pi}{n}\cos\frac{2\pi}{n}\cos\frac{3\pi}{n}$. One can verify that $f(1) = 1, f(2) = 0, f(3) = \frac{1}{4}, f(4) = 0, f(5) = -\cos^2\frac{2\pi}{5}\cos\frac{\pi}{5} < 0, f(6) = 0$ and $f(8) = \frac{1}{4}$. We shall show that $f(7) = \frac{1}{8}$.

Let ABC be an isosceles triangle with $\angle A = \frac{\pi}{7}$, $\angle B = \angle C = \frac{3\pi}{7}$, $BC = 1$ and $AB = AC = x$. Let D be the point on AC such that $\angle CBD = \frac{2\pi}{7}$. Let $BD = y$. Then the triangles BCD and ADB are isosceles with $BC = CD = 1$ and $AD = BD = y$. Thus $\cos\frac{\pi}{7} = \cos A = \frac{x}{2y}$, $\cos\frac{2\pi}{7} = \cos\angle CBD = \frac{y}{2}$, and $\cos\frac{3\pi}{7} = \cos C = \frac{1}{2x}$. Therefore, $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$.



Lastly, let's show that $f(n) \neq \frac{1}{n+1}$ for $n \geq 9$. For $n \geq 9$, we have $0 < \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n} < \frac{\pi}{2}$. Since cosine is a decreasing function on $[0, \frac{\pi}{2}]$, we have $f(n)$ is an increasing function of n for $n \geq 9$. Consequently, $f(n) \geq f(9) > \cos^3\frac{3\pi}{9} = \frac{1}{8} > \frac{1}{n+1}$.

4. Let $a_i = \max S_i$. Without loss of generality, assume that $a_1 \leq a_i$ for all i. We shall prove by induction on k. For $k = 2$, since $S_1 \cap S_2 \neq \emptyset$, $a_1 \in S_2$. Therefore $X = \{a_1\}$ works. Now assume that the result is true for $k - 1$. Let $\mathbb{I}$ be the collection consisting of S_1 and the sets S_i such that $S_i \cap S_1 \neq \emptyset$ and let $\mathbb{J}$ be the collection of the other sets. Note that a_1 is contained in all the sets in $\mathbb{I}$. If $|\mathbb{J}| < k - 1$, then the set X consisting of one integer from each of the sets in $\mathbb{J}$ together with a_1 has the desired property. Otherwise, consider a collection $\mathbb{K}$ of $k - 1$ sets in $\mathbb{J}$. $\mathbb{K}$, together with S_1, forms a collection of k sets. Among these there are two that have nonempty intersection. Since S_1 does not intersect any of the sets in $\mathbb{J}$, these two sets must come from $\mathbb{K}$. Thus by the induction hypothesis, there is a set X' of $k - 2$ integers such that every set in $\mathbb{J}$ contains one integer in X'. Thus $X = X' \cup \{a_1\}$ has the desired property.

5. Dividing each of the numerator and denominator of LHS by $2x_1x_2$, $2x_2x_3$, ..., writing $a_1 = \frac{x_3x_4}{x_1x_2}$, $a_2 = \frac{x_4x_5}{x_2x_3}$, ..., and noting that $x_i^2 + x_{i+1}^2 \geq 2x_ix_{i+1}$, we get

$$2 \times \text{LHS} \leq \frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n}.$$

Note that $a_1 a_2 \cdots a_n = 1$. It suffices to show that

$$\frac{a_1}{1+a_1} + \frac{a_2}{1+a_2} + \cdots + \frac{a_n}{1+a_n} \geq 1 \tag{$*$}$$

since it is equivalent to

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n} \leq n-1.$$

We shall show that $(*)$ is true for $n \geq 2$. The case $n = 2$ is obvious. We will now prove it by induction. Suppose $(*)$ holds for $n = k$. Now assume $a_1 \cdots a_{k+1} = 1$, $a_i > 0$ for all i. To prove the inductive step, it suffices to show that

$$\frac{a_k}{1+a_k} + \frac{a_{k+1}}{1+a_{k+1}} \geq \frac{a_k a_{k+1}}{1+a_k a_{k+1}}.$$

which can be verified directly.

Note: This is an extension of the problem :

$$\frac{x_1^2}{x_1^2+x_2x_3} + \frac{x_2^2}{x_2^2+x_3x_4} + \cdots + \frac{x_{n-1}^2}{x_{n-1}+x_nx_1} + \frac{x_n^2}{x_n^2+x_1x_2} \leq n-1.$$

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2011
(Open Section, Round 1)

Wednesday, 1 June 2011 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Throughout this paper, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$ (This notation is used in Questions 7, 9, 19 and 20).

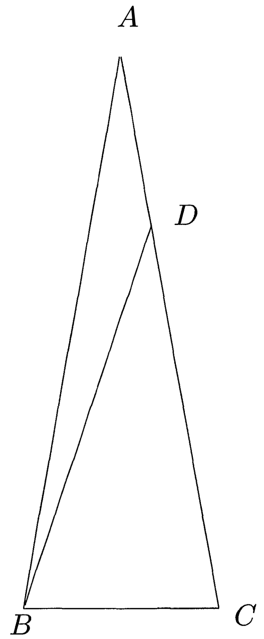
1. A circular coin A is rolled, without sliding, along the circumference of another stationary circular coin B with radius twice the radius of coin A. Let x be the number of degrees that the coin A makes around its centre until it first returns to its initial position. Find the value of x.

2. Three towns X, Y and Z lie on a plane with coordinates $(0,0)$, $(200,0)$ and $(0,300)$ respectively. There are 100, 200 and 300 students in towns X, Y and Z respectively. A school is to be built on a grid point (x, y), where x and y are both integers, such that the overall distance travelled by all the students is minimized. Find the value of $x+y$.

3. Find the last non-zero digit in 30!.

   (For example, $5! = 120$; the last non-zero digit is 2.)

4. The diagram below shows $\triangle ABC$, which is isoceles with $AB = AC$ and $\angle A = 20^\circ$. The point D lies on AC such that $AD = BC$. The segment BD is constructed as shown. Determine $\angle ABD$ in degrees.



5. Given that $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$, evaluate $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha}$.

6. The number 25 is expressed as the sum of positive integers $x_1, x_2, \cdots, x_k$, where $k \le 25$. What is the maximum value of the product of $x_1, x_2, x_3, \cdots$, and x_k?

7. Let x_0 be the largest (real) root of the equation $x^4 - 16x - 12 = 0$. Evaluate $\lfloor 10x_0 \rfloor$.

8. Let $x_i \in \{\sqrt{2}-1, \sqrt{2}+1\}$, where $i = 1, 2, 3, \cdots, 2012$. Define

$$S = x_1x_2 + x_3x_4 + x_5x_6 + \cdots + x_{2009}x_{2010} + x_{2011}x_{2012}.$$

How many different positive integer values can S attain?

9. Let A be the set of real numbers x satisfying the inequality $x^2 + x - 110 < 0$ and B be the set of real numbers x satisfying the inequality $x^2 + 10x - 96 < 0$. Suppose that the set of integer solutions of the inequality $x^2 + ax + b < 0$ is exactly the set of integers contained in $A \cap B$. Find the maximum value of $\lfloor |a - b| \rfloor$.

10. Given that

$$\begin{aligned} \alpha + \beta + \gamma &= 14 \\ \alpha^2 + \beta^2 + \gamma^2 &= 84 \\ \alpha^3 + \beta^3 + \gamma^3 &= 584, \end{aligned}$$

find $\max\{\alpha, \beta, \gamma\}$.

11. Determine the largest even positive integer which cannot be expressed as the sum of two composite odd positive integers.

12. Let a, b, c be positive integers such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ and $\gcd(a, b, c) = 1$. Suppose $a + b \leq 2011$. Determine the largest possible value of $a + b$.

13. Let $x[n]$ denote $x^{x^{\cdot^{\cdot^{x}}}}$, where there are n terms of x. What is the minimum value of n such that $9[9] < 3[n]$?

(For example, $3[2] = 3^3 = 27$; $2[3] = 2^{2^2} = 16$.)

14. In the triangle ABC, $\angle B = 90^\circ$, $\angle C = 20^\circ$, D and E are points on BC such that $\angle ADC = 140^\circ$ and $\angle AEC = 150^\circ$. Suppose $AD = 10$. Find $BD \cdot CE$.

15. Let $S = \{1, 2, 3, \cdots, 65\}$. Find the number of 3-element subsets $\{a_1, a_2, a_3\}$ of S such that $a_i \leq a_{i+1} - (i + 2)$ for $i = 1, 2$.

16. Determine the value of

$$\frac{3}{\sin^2 20^\circ} - \frac{1}{\cos^2 20^\circ} + 64\sin^2 20^\circ.$$

17. A real-valued function f satisfies the relation

$$f(x^2 + x) + 2f(x^2 - 3x + 2) = 9x^2 - 15x$$

for all real values of x. Find f(2011).

18. A collection of 2011 circles divide the plane into N regions in such a way that any pair of circles intersects at two points and no point lies on three circles. Find the last four digits of N.

19. If a positive integer N can be expressed as $\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor$ for some real numbers x, then we say that N is "visible"; otherwise, we say that N is "invisible". For example, 8 is visible since $8 = \lfloor 1.5 \rfloor + \lfloor 2(1.5) \rfloor + \lfloor 3(1.5) \rfloor$, whereas 10 is invisible. If we arrange all the "invisible" positive integers in increasing order, find the 2011^{th} "invisible" integer.

20. Let A be the sum of all non-negative integers n satisfying

$$\lfloor \frac{n}{27} \rfloor = \lfloor \frac{n}{28} \rfloor.$$

Determine A.

21. A triangle whose angles are $A,\ B,\ C$ satisfies the following conditions

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \frac{12}{7},$$

and

$$\sin A \sin B \sin C = \frac{12}{25}.$$

Given that $\sin C$ takes on three possible values $s_1,\ s_2$ and s_3, find the value of $100 s_1 s_2 s_3$.

22. Let $x > 1,\ y > 1$ and $z > 1$ be positive integers for which the following equation

$$1! + 2! + 3! + \ldots + x! = y^z$$

is satisfied. Find the largest possible value of $x + y + z$.

23. Let ABC be a non-isosceles acute-angled triangle with circumcentre O, orthocentre H and $\angle C = 41^\circ$. Suppose the bisector of $\angle A$ passes through the midpoint M of OH. Find $\angle HAO$ in degrees.

24. The circle γ_1 centred at O_1 intersects the circle γ_2 centred at O_2 at two points P and Q. The tangent to γ_2 at P intersects γ_1 at the point A and the tangent to γ_1 at P intersects $\gamma_2$at the point B where A and B are distinct from P. Suppose $PQ \cdot O_1O_2 = PO_1 \cdot PO_2$ and $\angle APB$ is acute. Determine the size of $\angle APB$ in degrees.

25. Determine

$$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{\binom{n}{i}}.$$

(Note: Here $\binom{n}{i}$ denotes $\dfrac{n!}{i!(n-i)!}$ for $i = 0, 1, 2, 3, \cdots, n$.)

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2011
(Open Section, Round 1 Solution)

1. **Answer.** 1080

   Solution. The number of complete revolutions the first coin A has turned through is the sum of two components: the number of revolutions round the stationary coin B if A were *sliding* on B and the number of revolutions round A's own axis (perpendicular to its plane and through its centre) determined by the distance travelled on the circumference of B. Thus, the total number of revolutions is

$$1 + \frac{2\pi(2r)}{2\pi r} = 3.$$

   Hence the number of degrees $= 3 \times 360 = 1080$. $\square$

2. **Answer.** 300

   Solution. We claim that the school must be built in Z. Suppose the school is to be built at another point A. The change in distance travelled

$$\begin{aligned} &= 300ZA + 200YA + 100XA - 200YZ - 100XZ \\ &= 100(ZA + AX - ZX) + 200(ZA + AY - ZY) \\ &> 0 \end{aligned}$$

   by triangle inequality. Thus, $\min(x+y) = 0 + 300 = 300$. $\square$

3. **Answer.** 8

   Solution. We first obtain the prime factorization of 30!. Observe that 29 is the largest prime number less than 30. We have

$$\begin{aligned} &\left\lfloor \frac{30}{2} \right\rfloor + \left\lfloor \frac{30}{2^2} \right\rfloor + \left\lfloor \frac{20}{2^3} \right\rfloor + \left\lfloor \frac{30}{2^4} \right\rfloor = 26 \\ &\left\lfloor \frac{30}{3} \right\rfloor + \left\lfloor \frac{30}{3^2} \right\rfloor + \left\lfloor \frac{30}{3^3} \right\rfloor = 14 \\ &\left\lfloor \frac{30}{5} \right\rfloor + \left\lfloor \frac{30}{5^2} \right\rfloor = 7 \\ &\left\lfloor \frac{30}{7} \right\rfloor = 4 \\ &\left\lfloor \frac{30}{11} \right\rfloor = 2 \\ &\left\lfloor \frac{30}{13} \right\rfloor = 2 \\ &\left\lfloor \frac{30}{17} \right\rfloor = \left\lfloor \frac{30}{19} \right\rfloor = \left\lfloor \frac{30}{23} \right\rfloor = \left\lfloor \frac{30}{29} \right\rfloor = 1. \end{aligned}$$

Thus,

$$\begin{aligned}
30! &= 2^{26}\cdot 3^{14}\cdot 5^7\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 29\\
\frac{30!}{10^7} &= 2^{19}\cdot 3^{14}\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 29\\
&= 6^{14}\cdot 2^5\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 29\\
&\equiv 6(2)(1)(1)(9)(7)(9)(3)(9) \pmod{10}\\
&\equiv 2(-1)(-3)(-1)(3)(-1) \pmod{10}\\
&\equiv 8 \pmod{10},
\end{aligned}$$

showing that the last non-zero digit is 8.

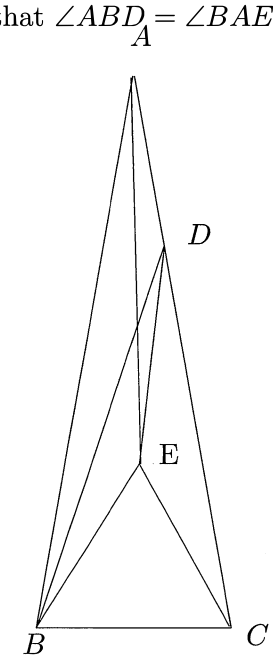
□

4. **Answer.** 10

Solution. Let E be the point inside $\triangle ABC$ such that $\triangle EBC$ is equilateral. Connect A and D to E respectively.

It is clear that $\triangle AEB$ and $\triangle AEC$ are congruent, since $AE = AE$, $AB = AC$ and $BE = CE$. It implies that $\angle BAE = \angle CAE = 10^o$.

Since $AD = BC = BE$, $\angle EBA = \angle DAB = 20^o$ and $AB = BA$, we have $\triangle ABE$ and $\triangle BAD$ are congruent, implying that $\angle ABD = \angle BAE = 10^o$.



□

5. **Answer.** 1

Solution. Since $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$, set $\cos\theta = \dfrac{\cos^2\alpha}{\cos\beta}$ and $\sin\theta = \dfrac{\sin^2\alpha}{\sin\beta}$. Then

$$\cos(\theta-\alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos^2\alpha + \sin^2\alpha = 1.$$

and so

$$\theta - \alpha = 2k\pi \text{ for some } k \in \mathbb{Z}.$$

Thus $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha$. Consequently,

$$\frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = \cos^2\beta + \sin^2\beta = 1.$$

$\square$

6. **Answer.** 8748

   Solution. Clearly, no x_i should be 1. If $x_i \geq 4$, then splitting it into two factors 2 and $x_i - 2$ will give a product of $2x_i - 4$ which is at least as large as x_i. Further, $3\times 3 > 2\times 2\times 2$, so any three factors of 2 should be replaced by two factors of 3. Thus, split 25 into factors of 3, retaining two 2's, which means $25 = 7\times 3 + 2\times 2$. The maximum product is thus $3^7 2^2 = 8748$. $\square$

7. **Answer.** 27

   Solution. Since $x^4 - 16x - 12 \equiv x^4 + 4x^2 + 4 - 4(x^2 + 4x + 4) \equiv (x^2 - 2x - 2)(x^2 + 2x + 6)$, we conclude that $x_0 = 1 + \sqrt{3}$ and so $1 + \sqrt{2.89} < x_0 < 1 + \sqrt{3.24}$. Consequently, $\lfloor 10x_0 \rfloor = 27$. $\square$

8. **Answer.** 504

   Solution. Note that $(\sqrt{2}-1)^2 = 3 - 2\sqrt{2}$, $(\sqrt{2}+1)^2 = 3 + 2\sqrt{2}$ and $(\sqrt{2}-1)(\sqrt{2}+1) = 1$. There are 1006 pairs of products in S; each pair of the product can be either $3-2\sqrt{2}, 3+2\sqrt{2}$ or 1. Let a be the number of these products with value $3 - 2\sqrt{2}$, b be the number of these products with value $3+2\sqrt{2}$ and c be the number of them with value 1. The $a+b+c = 1006$. Hence

   $$S = a(3 - 2\sqrt{2}) + b(3 + 2\sqrt{2}) + c = 3a + 3b + c + 2\sqrt{2}(b - a).$$

   For S to be a positive integer, $b = a$ and thus $2a + c = 1006$. Further,

   $$S = 6a + c = 6a + 1006 - 2a = 4a + 1006.$$

   From $2a + c = 1006$ and that $0 \leq a \leq 503$, it is clear that S can have 504 different positive integer values. $\square$

9. **Answer.** 71

   Note that $x^2 + x - 110 = (x - 10)(x + 11)$. Thus the set of real numbers x satisfying the inequality $x^2 + x - 110 < 0$ is $-11 < x < 10$.

   Also note that $x^2 + 10x - 96 = (x - 6)(x + 16)$. Thus the set of real numbers x satisfying the inequality $x^2 + 10x - 96 < 0$ is $-16 < x < 6$.

   Thus $A = \{x : -11 < x < 10\}$ and $B = \{x : -16 < x < 6\}$, implying that

   $$A \cap B = \{x : -11 < x < 6\}.$$

Now let $x^2 + ax + b = (x - x_1)(x - x_2)$, where $x_1 \le x_2$. Then the set of integer solutions of $x^2 + ax + b < 0$ is

$$\{k : k \text{ is an integer}, x_1 < k < x_2\}.$$

By the given condition,

$$\{k : k \text{ is an integer}, x_1 < k < x_2\} = \{k : k \text{ is an integer}, -11 < k < 6\}$$
$$= \{-10, -9, \cdots, 5\}.$$

Thus $-11 \le x_1 < -10$ and $5 < x_2 \le 6$. It implies that $-6 < x_1 + x_2 < -4$ and $-66 \le x_1 x_2 < -50$.

From $x^2 + ax + b = (x - x_1)(x - x_2)$, we have $a = -(x_1 + x_2)$ and $b = x_1 x_2$. Thus $4 < a < 6$ and $-66 \le b < -50$. It follows that $54 < a - b < 72$.

Thus $\max\lfloor |a - b| \rfloor \le 71$.

It remains to show that it is possible that $\lfloor |a - b| \rfloor = 71$ for some a and b.

Let $a = 5$ and $b = -66$. Then $x^2 + ab + b = (x + 11)(x - 6)$ and the inequality $x^2 + ab + b < 0$ has solutions $\{x : -11 < x < 6\}$. So the set of integer solutions of $x^2 + ab + b < 0$ is really the set of integers in $A \cap B$.

Hence $\max\lfloor |a - b| \rfloor = 71$. □

10. **Answer.** 8

Solution. We consider the polynomial

$$P(t) = t^3 + bt^2 + ct + d.$$

Suppose the root of the equation $P(t) = 0$ are x, y and z. Then

$$-b = x + y + z = 14,$$

$$c = xy + xz + yz = \frac{1}{2}\left((x + y + z)^2 - x^2 - y^2 - z^2\right) = \frac{1}{2}\left(14^2 - 84\right) = 56$$

and

$$x^3 + y^3 + z^3 + 3d = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz).$$

Solving for b, c and d, we get $b = -14$, $c = 30$ and $d = -64$. Finally, since $t^3 - 14t^2 + 30t - 64 = 0$ implies $t = 2$ or $t = 4$ or $t = 8$, we conclude that $\max\{\alpha, \beta, \gamma\} = 8$. □

11. **Answer.** 38

Solution. Let n be an even positive integer. Then each of the following expresses n as the sum of two odd integers: $n = (n - 15) + 15, (n - 25) + 25$ or $(n - 35) + 35$. Note that at least one of $n - 15, n - 25, n - 35$ is divisible by 3, hence n can be expressed as the sum of two composite odd numbers if $n > 38$. Indeed, it can be verified that 38 cannot be expressed as the sum of two composite odd positive integers. □

12. **Answer.** 1936

Solution. We first show that $a+b$ must be a perfect square. The equation $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$ is equivalent to $\frac{a-c}{c}=\frac{c}{b-c}$. Write $\frac{a-c}{c}=\frac{c}{b-c}=\frac{p}{q}$, where $\gcd(p,q)=1$. From $\frac{a-c}{c}=\frac{p}{q}$, we have $\frac{a}{p+q}=\frac{c}{q}$. Since $\gcd(p,q)=1$, we must have q divides c. Similarly from $\frac{b-c}{c}=\frac{q}{p}$, we have $\frac{b}{p+q}=\frac{c}{p}$. Since $\gcd(p,q)=1$, we must have p divides c. Thus $\gcd(p,q)=1$ implies pq divides c. Therefore $\frac{a}{p(p+q)}=\frac{b}{q(p+q)}=\frac{c}{pq}$ is an integer r. Then r divides a, b and c, so that $r=1$ since $\gcd(a,b,c)=1$. Consequently, $a+b=p(p+q)+q(p+q)=(p+q)^2$.

Next the largest square less than or equal to 2011 is $44^2=1936$. As $1936=1892+44$, and $\frac{1}{1892}+\frac{1}{44}=\frac{1}{43}$, where $\gcd(1892,44,43)=1$, we have $a=1892$, $b=44$ and $c=43$ give the largest value of $a+b$. These values of a,b,c can obtained from the identity $\frac{1}{m^2-m}+\frac{1}{m}=\frac{1}{m-1}$. □

13. **Answer.** 10

Solution. Suppose $9[m]<3[n]$. Note that $9[m]=3^p$ and $3[n]=3^q$ for some integers p and q. Thus, $q\geq p+1$. In particular,

$$2(9[m])<3(9[m])=3^{p+1}\leq 3^q=3[n].$$

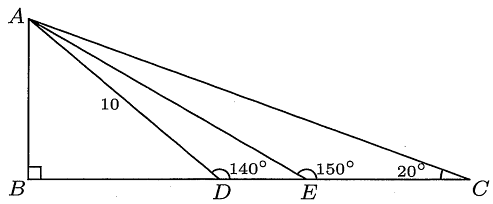
Then we have

$$9[m+1]=(3^2)^{9[m]}=3^{2(9[m])}<3^{3[n]}=3[n+1].$$

Thus, $9[m]<3[n]$ implies $9[m+1]<3[n+1]$. It is clear that $9[2]=81=3^4<3[3]$. Continuing this way, $9[9]<3[10]$. It is also clear that $9[9]>3[9]$, hence the minimum value of n is 10. □

14. **Answer.** 50

Direct calculation gives $\angle DAC=20^\circ$ and $\angle BAD=50^\circ$. Thus $AD=CD=10$. Also $BD=10\sin 50^\circ$. By sine rule applied to the triangle AEC, we have $\frac{CE}{\sin 10^\circ}=\frac{AC}{\sin 150^\circ}=\frac{2\times 10\cos 20^\circ}{\sin 150}=40\cos 20^\circ$. (Note that $AD=DC$.)



Therefore, $BD\cdot CE=400\cos 20^\circ\sin 10^\circ\sin 50^\circ$.

Direct calculation shows that $\cos 20^\circ\sin 10^\circ\sin 50^\circ=\frac{1}{8}$ so that $BD\cdot CE=50$. □

15. **Answer.** 34220

Solution. Note that the condition $a_i\leq a_{i+1}-(i+2)$ for $i=1,2$ is equivalent to that

$$a_1+3\leq a_2,\quad a_2+4\leq a_3.$$

Let A be the set of all 3-element subsets $\{a_1,a_2,a_3\}$ of S such that $a_1+3\leq a_2$ and $a_2+4\leq a_3$.

Let B be the set of all 3-element subsets $\{b_1,b_2,b_3\}$ of the set $\{1,2,\cdots,60\}$.

We shall show that $|A| = |B| = \binom{60}{3} = 34220$ by showing that the mapping ϕ below is a bijection from A to B:

$$\phi : \{a_1, a_2, a_3\} \longrightarrow \{a_1, a_2 - 2, a_3 - 5\}.$$

First, since $\{a_1, a_2, a_3\} \in A$, we have $a_1+3 \leq a_2$ and $a_2+4 \leq a_3$, and so $a_1 < a_2-2 < a_3-5$, implying that $\{a_1, a_2 - 2, a_3 - 5\} \in B$.

It is clear that ϕ is injective.

It is also surjective, as for any $\{b_1, b_2, b_3\} \in B$ with $b_1 < b_2 < b_3$, we have $\{b_1, b_2+2, b_3+5\} \in A$ and

$$\phi : \{b_1, b_2 + 2, b_3 + 5\} \longrightarrow \{b_1, b_2, b_3\}.$$

Hence ϕ is a bijection and $|A| = |B| = 34220$. □

16. **Answer.** 32

Solution. It is clear that $8(\cos 40°)^3 - 6\cos 40° + 1 = 0$, since $\cos 3A = 4\cos^3 A - 3\cos A$. Observe that

$$\begin{aligned}
& \frac{3}{\sin^2 20°} - \frac{1}{\cos^2 20°} + 64\sin^2 20° \\
= & \frac{6}{1 - \cos 40°} - \frac{2}{1 + \cos 40°} + 32(1 - \cos 40°) \\
= & \frac{8\cos 40° + 4}{1 - (\cos 40°)^2} + 32 - 32\cos 40° \\
= & \frac{8\cos 40° + 4 - 32\cos 40° + 32(\cos 40°)^3}{1 - (\cos 40°)^2} + 32 \\
= & 4 \times \frac{1 - 6\cos 40° + 8(\cos 40°)^3}{1 - (\cos 40°)^2} + 32 \\
= & 32,
\end{aligned}$$

where the last step follows from $8(\cos 40°)^3 - 6\cos 40° + 1 = 0$. □

17. **Answer.** 6029

Solution. Given the original equation

$$f(x^2 + x) + 2f(x^2 - 3x + 2) = 9x^2 - 15x,$$

we replace x by $1 - x$ and obtain

$$f(x^2 - 3x + 2) + 2f(x^2 + x) = 9(1 - x)^2 - 15(1 - x) = 9x^2 - 3x - 6.$$

Eliminating $f(x^2 - 3x + 2)$ from the two equations, we obtain

$$3f(x^2 + x) = 9x^2 + 9x - 12,$$

thereby

$$f(x^2 + x) = 3x^2 + 3x - 4 = 3(x^2 + x) - 4,$$

hence $f(2011) = 3(2011) - 4 = 6029$.

□

18. **Answer.** 2112

Solution. We denote the numbers of regions divided by n circles by $P(n)$. We have $P(1)=2,\ P(2)=4,\ P(3)=8,\ P(4)=14,\dots$ and from this we notice that

$$\begin{aligned}
P(1) &= 2,\\
P(2) &= P(1)+2,\\
P(3) &= P(2)+4,\\
P(4) &= P(3)+6,\\
\dots & \quad \dots\\
P(n) &= P(n-1)+2(n-1).
\end{aligned}$$

Summing these equations, we obtain

$$P(n) = 2+2+4+\ldots+2(n-1) = 2+n(n-1).$$

This formula can be shown by induction on n to hold true.
Base case: $n=1$ is obvious.
Inductive step: Assume that the formula holds for $n=k\geq 1$, i.e., $P(k)=2+k(k-1)$. Consider $k+1$ circles, the $(k+1)$-th circle intersects k other circles at $2k$ points (for each one, it cuts twice), which means that this circle is divided into $2k$ arcs, each of which divides the region it passes into two sub-regions. Therefore, we have in addition $2k$ regions, and so

$$P(k+1) = P(k)+2k = 2+k(k-1)+2k = 2+k(k+1).$$

The proof by induction is thus complete.
Using this result, put $n=2011$, the number of regions $N = 2+2011\cdot(2011-1) = 4042112$. So, the last 4 digits are 2112. $\square$

19. **Answer.** 6034

Solution. Let n be a positive integer.
If $n \leq x < n+\frac{1}{3}$, then $2n \leq 2x < 2n+\frac{2}{3}$ and $3n \leq 3x < 3n+1$, giving

$$N = \lfloor x\rfloor + \lfloor 2x\rfloor + \lfloor 3x\rfloor = n+2n+3n = 6n.$$

If $n+\frac{1}{3} \leq x < n+\frac{1}{2}$, then $2n+\frac{2}{3} \leq 2x < 2n+1$ and $3n+1 \leq 3x < 3n+\frac{3}{2}$, giving

$$N = \lfloor x\rfloor + \lfloor 2x\rfloor + \lfloor 3x\rfloor = n+2n+3n+1 = 6n+1.$$

If $n+\frac{1}{2} \leq x < n+\frac{2}{3}$, then $2n+1 \leq 2x < 2n+\frac{4}{3}$ and $3n+\frac{3}{2} \leq 3x < 3n+\frac{4}{3}$, giving

$$N = \lfloor x\rfloor + \lfloor 2x\rfloor + \lfloor 3x\rfloor = n+2n+1+3n+1 = 6n+2.$$

If $n+\frac{2}{3} \leq x < n+1$, then $2n+\frac{4}{3} \leq 2x < 2n+2$ and $3n+2 \leq 3x < 3n+3$, giving

$$N = \lfloor x\rfloor + \lfloor 2x\rfloor + \lfloor 3x\rfloor = n+2n+1+3n+2 = 6n+3.$$

Thus, "invisible" numbers must be of the form $6n+4$ and $6n+5$. The 2011^{th} "invisible" integer is $4+6\times\dfrac{2011-1}{2} = 6034$. $\square$

20. **Answer.** 95004

Solution. We shall prove that for any positive integer a, if $f(a)$ denotes the sum of all nonnegative integer solutions to $\lfloor \frac{n}{a} \rfloor = \lfloor \frac{n}{a+1} \rfloor$, then

$$f(a) = \frac{1}{6}a(a^2-1)(a+2).$$

Thus $f(27) = 95004$.

Let n be a solution to $\lfloor \frac{n}{a} \rfloor = \lfloor \frac{n}{a+1} \rfloor$. Write $n = aq + r$, where $0 \le r < a$. Thus $\lfloor \frac{n}{a} \rfloor = q$. Also $n = (a+1)q + r - q$. Since $\lfloor \frac{n}{a+1} \rfloor = q$, we have $0 \le r - q$, that is, $q \le r < a$. Therefore for each $q = 0, 1, \dots, a-1$, r can be anyone of the values $q, q+1, \dots, a-1$. Thus

$$\begin{aligned}
A &= \sum_{q=0}^{a-1}\sum_{r=q}^{a-1}(qa+r) \\
&= \sum_{q=0}^{a-1}(a-q)qa + \sum_{q=0}^{a-1}\sum_{r=q}^{a-1} r \\
&= a^2\sum_{q=0}^{a-1} q - a\sum_{q=0}^{a-1} q^2 + \sum_{r=0}^{a-1}\sum_{q=0}^{r} r \\
&= a^2\sum_{q=0}^{a-1} q - a\sum_{q=0}^{a-1} q^2 + \sum_{r=0}^{a-1} r(r+1) \\
&= a^2\sum_{q=0}^{a-1} q - a\sum_{q=0}^{a-1} q^2 + \sum_{r=0}^{a-1} r^2 + \sum_{r=0}^{a-1} r \\
&= (a^2+1)\cdot\frac{1}{2}a(a-1) + (1-a)\cdot\frac{1}{6}a(2a-1)(a-1) \\
&= \frac{1}{6}a(a^2-1)(a+2).
\end{aligned}$$

□

21. **Answer.** 48

By using factor formulae and double angle formulae:

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{12}{7},$$

and

$$\sin A \sin B \sin C = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{12}{25}.$$

Solving these equations, we obtain

$$\begin{aligned}
\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &= 0.1 \\
\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} &= 0.6
\end{aligned}$$

Furthermore,

$$\sin\frac{C}{2} = \cos\left(\frac{A+B}{2}\right) = \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2},$$

multiplying both sides by $\sin\frac{C}{2}\cos\frac{C}{2}$, we get

$$\sin^2\frac{C}{2}\cos\frac{C}{2} = 0.6\sin\frac{C}{2} - 0.1\cos\frac{C}{2}.$$

or equivalently,

$$(1-t^2)t = 0.6\sqrt{1-t^2} - 0.1t \iff 11t - 10t^3 = 6\sqrt{1-t^2},$$

where $t = \cos\frac{C}{2}$. This equation solves for $t = \sqrt{\frac{1}{2}},\ \sqrt{\frac{4}{5}},\ \sqrt{\frac{3}{10}}$, and so the corresponding values of $\sin C$ are

$$1,\ 0.8,\ 0.6$$

and hence $100s_1s_2s_3 = 100\cdot 1\cdot 0.8\cdot 0.6 = 48.$ $\square$

22. **Answer.** 8

Solution. We first prove that if $x \geq 8$, then $z = 2$. To this end, we observe that the left hand side of the equation $1! + 2! + 3! + \ldots + x!$ is divisible by 3, and hence $3 \mid y^z$. Since 3 is a prime, $3 \mid y$. So, $3^z \mid y^z$ by elementary properties of divisibility.

On the other hand, when $x = 8$,

$$1! + 2! + \ldots + 8! = 46233$$

is divisible by 3^2 but not by 3^3. Now, note that if $n \geq 9$, then we have $3^3 \mid n!$. So, when $x \geq 8$, the left hand side is divisible by 3^2 but not by 3^3. This means that $z = 2$.

We now prove further that when $x \geq 8$, then the given equation has no solutions. To prove this, we observe that $x \geq 8$ implies that

$$1! + 2! + 3! + 4! + \underbrace{5! + \ldots x!}_{\text{divisible by } 5} \equiv 3 \pmod 5.$$

Since we have deduced that $z = 2$, we only have $y^2 \equiv 0,\ 1,\ -1 \pmod 5$. This mismatch now completes the argument that there are no solutions to the equation when $x \geq 8$.

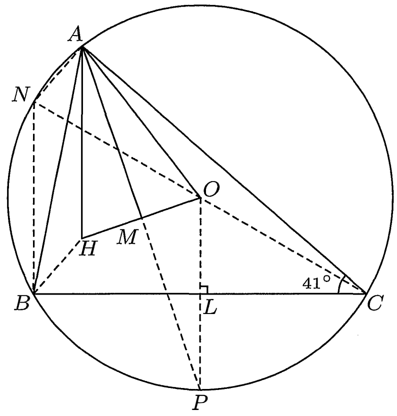
So the search narrows down to $x < 8$. By exhaustion, it is easy to find that there is only one solution:

$$x = y = 3,\ z = 2.$$

Thus, the sum of this only combination must be the largest and is equal to $3 + 3 + 2 = 8$. $\square$

23. **Answer.** 38

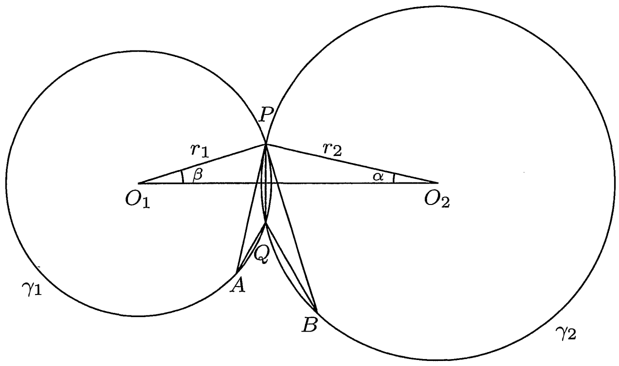
Let P be the midpoint of the arc BC not containing A on the circumcircle of the triangle ABC. Then OP is the perpendicular bisector of BC. Since AM bisects $\angle A$, the points A, M, P are collinear. As both AH and OP are perpendicular to BC, they are parallel. Thus $\angle HAM = \angle OPM = \angle OAM$. Also $\angle HMA = \angle OMP$. Since $HM = OM$, we have the triangles AHM and POM are congruent. Therefore $AH = PO = AO$.



Let L be the midpoint of BC. It is a known fact that $AH = 2OL$. To see this, extend CO meeting the circumcircle of the triangle ABC at the point N. Then $ANBH$ is a parallelogram. Thus $AH = NB = 2OL$. Therefore in the right-angled triangle OLC, $OC = OA = AH = 2OL$. This implies $\angle OCL = 30°$. Since the triangle ABC is acute, the circumcentre O lies inside the triangle. In fact $\angle A = 60°$ and $\angle B = 79°$. Then $\angle OAC = \angle OCA = 41° - 30° = 11°$. Consequently, $\angle HAO = 2\angle OAM = 2 \times (30° - 11°) = 38°$. $\square$

24. **Answer.** 30

Let $PO_1 = r_1$ and $PO_2 = r_2$. First note that O_1O_2 intersects PQ at the midpoint H (not shown in the figure) of PQ perpendicularly. Next observe that $\angle APQ = \angle PBQ = \angle PO_2O_1$, and $\angle BPQ = \angle PAQ = \angle PO_1O_2$. Therefore $\angle APB = \angle APQ + \angle BPQ = \angle PO_2O_1 + \angle PO_1O_2$.



Let $\angle PO_2O_1 = \alpha$ and $\angle PO_1O_2 = \beta$. Then $\sin\alpha = \frac{PQ}{2r_2}, \cos\alpha = \frac{O_2H}{r_2}$ and $\sin\beta = \frac{PQ}{2r_1}, \cos\beta = \frac{O_1H}{r_1}$. Thus $\sin\angle APB = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{PQ}{2r_2}\cdot\frac{O_1H}{r_1} + \frac{O_2H}{r_2}\cdot\frac{PQ}{2r_1} = \frac{PQ\cdot(O_1H+O_2H)}{2r_1r_2} = \frac{PQ\cdot O_1O_2}{2r_1r_2} = \frac{1}{2}$. Since $\angle APB$ is acute, it is equal to $30°$. $\square$

25. **Answer.** 2

Solution. Let

$$a_n = \sum_{i=0}^{n} \binom{n}{i}^{-1}.$$

Assume that $n \geq 3$. It is clear that

$$a_n = 2 + \sum_{i=1}^{n-1} \binom{n}{i}^{-1} > 2.$$

Also note that

$$a_n = 2 + 2/n + \sum_{i=2}^{n-2} \binom{n}{i}^{-1}.$$

Since $\binom{n}{i} \geq \binom{n}{2}$ for all i with $2 \leq i \leq n-2$,

$$a_n \leq 2 + 2/n + (n-3)\binom{n}{2}^{-1} \leq 2 + 2/n + 2/n = 2 + 4/n.$$

So we have show that for all $n \geq 3$,

$$2 < a_n \leq 2 + 4/n.$$

Thus

$$\lim_{n\to\infty} a_n = 2.$$

$\square$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Open Section, Round 2)

Saturday, 2 July 2011 **0900-1330**

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1. In the acute-angled non-isosceles triangle ABC, O is its circumcentre, H is its orthocentre and $AB > AC$. Let Q be a point on AC such that the extension of HQ meets the extension of BC at the point P. Suppose $BD = DP$, where D is the foot of the perpendicular from A onto BC. Prove that $\angle ODQ = 90^\circ$.

2. If 46 squares are colored red in a 9×9 board, show that there is a 2×2 block on the board in which at least 3 of the squares are colored red.

3. Let $x, y, z > 0$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{1}{xyz}$. Show that

$$\frac{2x}{\sqrt{1+x^2}} + \frac{2y}{\sqrt{1+y^2}} + \frac{2z}{\sqrt{1+z^2}} < 3.$$

4. Find all polynomials $P(x)$ with real coefficients such that

$$P(a) \in \mathbb{Z} \quad \text{implies that} \quad a \in \mathbb{Z}.$$

5. Find all pairs of positive integers (m, n) such that

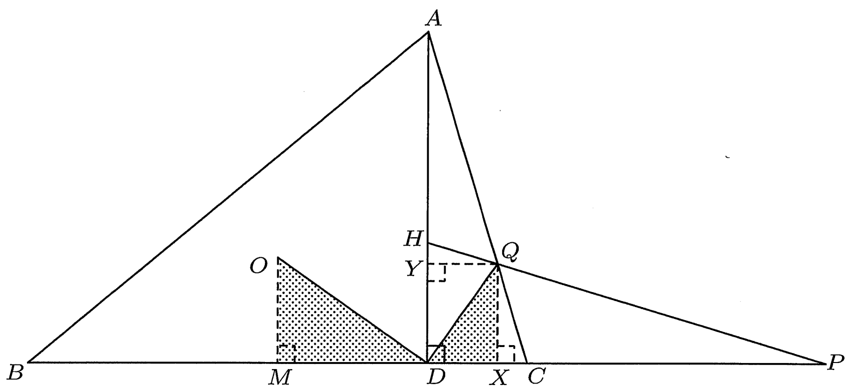
$$m + n - \frac{3mn}{m+n} = \frac{2011}{3}.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Open Section, Round 2 solutions)

1. Drop perpendiculars OM and QX onto BC, and QY from Q onto AD. First $2DM = DM + BD - BM = BD - (BM - DM) = PD - (CM - DM) = PD - CD = PC$. It is a well-known fact that $2OM = AH$.



Next $\angle CPQ = \angle DBH = \angle HAQ$ so that the triangles CPQ and HAQ are similar. Thus the triangles XPQ and YAQ are similar. Therefore

$$\frac{QX}{DX} = \frac{QX}{QY} = \frac{PC}{AH} = \frac{DM}{OM}.$$

Hence the triangles DXQ and OMD are similar. It follows that $\angle ODQ = 90°$.

2. Suppose that at most 2 squares are colored red in any 2×2 square. Then in any 9×2 block, there are at most 10 red squares. Moreover, if there are 10 red squares, then there must be 5 in each row. This can be seen as follows. There are 8 2×2 blocks. Counting multiplicity, there are altogether 16 red squares. Each red square in the interior is counted twice while each red square at the edge is counted once. If there are 11 red squares, then there are at least 7 red squres in the interior. Thus the total count is at least $4+7\times 2 = 18 > 16$, a contradiction. If there are exactly 10 red squares, then 4 of them must be at the edge and the red squares in each row are not next to each other and hence there 5 in each row.

Now let the number of red squares in row i be r_i. Then $r_i + r_{i+1} \leq 10$, $1 \leq i \leq 8$. Suppose that some $r_i \leq 5$ with i odd. Then

$$(r_1 + r_2) + \cdots + (r_{i-2} + r_{i-1}) + r_i + \cdots + (r_8 + r_9) \leq 4 \times 10 + 5 = 45$$

which leads to a contradiction. On the other hand, suppose that $r_1, r_3, r_5, r_7, r_9 \geq 6$. Then the sum of any 2 consecutive r_i's is ≤ 9. Again we get a contradiction as

$$(r_1 + r_2) + \cdots + (r_7 + r_8) + r_9 \leq 4 \times 9 + 9 = 45.$$

3. Let $r = 1/x, s = 1/y, t = 1/z$. There exists $\alpha < 1$ such that $r + s + t = \alpha^2 rst$ or $\alpha(r + s + t) = \alpha^3 rst$. Let $a = \alpha r$, $b = \alpha s$, $c = \alpha t$. Write $a = \tan A$, $b = \tan B$, $c = \tan C$, then $A + B + C = \pi$. It is clear that

$$\begin{aligned}\frac{1}{2} \times \text{LHS} &= \frac{1}{\sqrt{1+r^2}} + \frac{1}{\sqrt{1+s^2}} + \frac{1}{\sqrt{1+t^2}} \\ &< \frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \\ &= \cos A + \cos B + \cos B \\ &\leq 3\cos\left(\frac{A+B+C}{3}\right) = \frac{3}{2} = \frac{1}{2} \times \text{RHS}.\end{aligned}$$

2nd soln: Note that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{1}{xyz} \quad \Rightarrow \quad xy + yz + xz < 1.$$

Hence

$$\frac{2x}{\sqrt{1+x^2}} < \frac{2x}{\sqrt{x^2 + xy + xz + yz}} = \frac{2x}{\sqrt{(x+y)(x+z)}}.$$

By AM-GM we have

$$\frac{2x}{\sqrt{(x+y)(x+z)}} \leq \frac{x}{x+y} + \frac{x}{x+z}.$$

Similarly,

$$\frac{2y}{\sqrt{(y+z)(y+x)}} \leq \frac{y}{y+z} + \frac{y}{y+x}, \quad \frac{2z}{\sqrt{(z+x)(z+y)}} \leq \frac{z}{z+x} + \frac{z}{z+y}.$$

The desired inequality then follows by adding up the three inequalities.

4. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$. Define $Q(x) = P(x+1) - P(x)$. Then $Q(x)$ is of degree $n - 1$. We'll prove by contradiction that $|Q(x)| \leq 3$ for all x. This will imply that $n \leq 1$. Assume that $|Q(a)| > 3$ for some $a \in \mathbb{R}$. Then $|P(a+1) - P(a)| > 3$. Thus there are 3 integers between $P(a)$ and $P(a+1)$. Hence there exists three values of $x \in [a, a+1]$ such that $P(x)$ is an integer. Thus there are three integers in $[a, a+1]$, a contradiction. There are two cases:

Case (i) $n = 0$. Here we have $P(x) = c$ where $c \notin \mathbb{Z}$.

Case (ii) $n = 1$. Here $P(x) = sx + t$. There are two integers m, n such that $P(m) = sm + t = 0$ and $P(n) = sn + t = 1$. Thus $s(n - m) = 1$ implying that $1/s \in \mathbb{Z}$ and $sm + t = 0$ implying that $t/s \in \mathbb{Z}$. Letting $1/s = p$ and $t/s = q$, $P(x) = \frac{x}{p} + \frac{q}{p}$ where $p, q \in \mathbb{Z}$ and $p \neq 0$.

5. Answer: $(m, n) = (1144, 377)$ or $(377, 1144)$.

Let m and n be positive integers satisfying the given equation. That is $2011(m + n) = 3(m^2 - mn + n^2)$. Since the equation is symmetric in m and n, we may assume $m \geq n$. If $m = n$, then $m = n = 4022/3$ which is not an integer. So we may further assume $m > n$. Let $p = m+n$ and $q = m-n > 0$. Then $m = (p+q)/2$ and $n = (p-q)/2$, and the equation becomes $8044p = 3(p^2 + 3q^2)$. Since 3 does not divide 8044, it must divide p. By letting $p = 3r$, the above equation reduces to $8044r = 3(3r^2 + q^2)$. From this, 3 must divide r. By letting $r = 3s$, we get $8044s = 27s^2 + q^2$, or equivalently

$$s(8044 - 27s) = q^2. \tag{$*$}$$

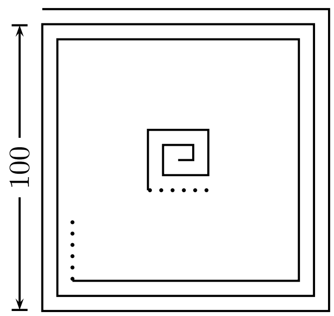
For s between 1 and $\lfloor 8044/27 \rfloor = 297$, the number $s(8044 - 27s)$ is a square only when $s = 169$. To narrow down the values of s, we proceed as follow.

Let $s = 2^\alpha u$, where α is a nonnegative integer and u is an odd positive integer. Suppose α is odd and $\alpha \geq 3$. Then $(*)$ becomes $2^{\alpha+2}u(2011 - 27 \times 2^{\alpha-2}u) = q^2$ which is a square. Since $\alpha + 2$ is odd, 2 must divide $2011 - 27 \times 2^{\alpha-2}u$ implying 2 divides 2011 which is a contradiction. Next suppose $\alpha = 1$. Then we have $u(2 \times 2011 - 27u) = (q/2)^2$. If u is not a square, then there exists an odd prime factor t of u such that t divides $2 \times 2011 - 27u$. Thus t divides 2×2011 so that t must be 2011 since 2011 is a prime. But then $u \geq t = 2011$ contradicting $2 \times 2011 - 27u > 0$. Therefore u must be a square. This implies that $2 \times 2011 - 27u$ is also a square. Taking mod 4, we have $u \equiv 0$ or $1 \pmod 4$ so that $2 \times 2011 - 27u \equiv 2$ or $3 \pmod 4$ which contradicts the fact that $2 \times 2011 - 27u$ is a square. Thus $\alpha \neq 1$ too. Consequently α must be even. Then dividing both sides of $(*)$ by 2^α, we obtain $u(8044 - 27 \times 2^\alpha u) = q^2/2^\alpha$ which is a square. Now suppose u is not a square. Then there exists an odd prime factor v of u such that v divides $8044 - 27 \times 2^\alpha u$. Then v must divide 8044 so that $v = 2011$. Thus $u \geq v = 2011$. This again contradicts the fact that $8044 - 27 \times 2^\alpha u > 0$. Therefore u is a square. Consequently s is also a square. Write $s = w^2$. Then $(*)$ becomes $w^2(8044 - 27w^2) = q^2 \geq 0$. From this $w \leq \lfloor (8044/27)^{\frac{1}{2}} \rfloor = 17$. A direct verification shows that $8044 - 27w^2$ is a square only when $w = 13$. Thus $s = w^2 = 169$. Then $p = 3r = 9s = 1521$, and by $(*)$ $q = (169 \times (8044 - 27 \times 169))^{\frac{1}{2}} = 767$. Lastly, $m = (p+q)/2 = 1144$ and $n = (p-q)/2 = 377$.

Errata for the 2011 SMO Solution Book

Updated: 24 April 2012

p7. Q28.



p12. Q16. Answer: 7981.

The total number of solutions is

$$\begin{aligned} & 3(1\times 2+2\times 3+\cdots+19\times 20)+1 \\ =\ & (2^3-1^3)+(3^3-2^3)+\cdots+(20^3-19^3)-20+1 \\ =\ & 20^3-19=7981. \end{aligned}$$

p13. Q19. Answer: 256.

So $m=(5-1-2)^2=4$ and $M=(5+1+2)^2=64$. Thus, $m\times M=256$.

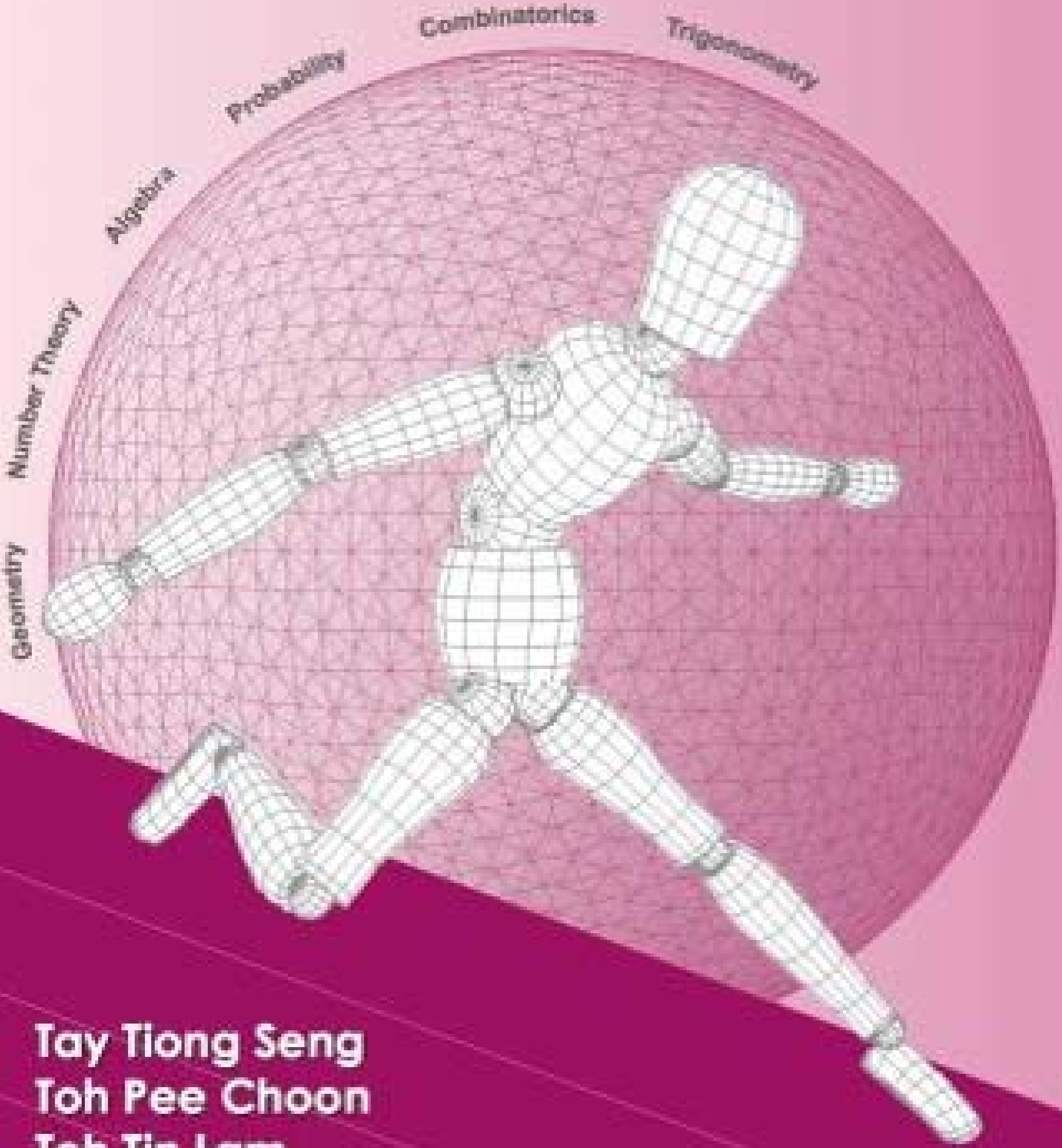
p16. Q26. Answer: 3.

There are 3 cases: $(4,1,1,1)$, $(3,2,1,1)$ and $(2,2,2,1)$.

Q28. Answer: 10403.

The broken line is constructed using "L", with lengths $2,4,6,\ldots,200,202$, and a segment of length 101. Then the total length is $2(1+2+3+\cdots+101)+101=10403$.

SINGAPORE MATHEMATICAL OLYMPIADS 2012



Tay Tiong Seng
Toh Pee Choon
Toh Tin Lam
Wang Fei

Published by Singapore Mathematical Society

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

Junior Section (First Round)

Tuesday, 29 May 2012 **0930–1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x. *For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Let α and β be the roots of the quadratic equation $x^2 + 2bx + b = 1$. The smallest possible value of $(\alpha - \beta)^2$ is

   (A) 0; (B) 1; (C) 2; (D) 3; (E) 4.

2. It is known that $n^{2012} + n^{2010}$ is divisible by 10 for some positive integer n. Which of the following numbers is not a possible value for n?

   (A) 2; (B) 13; (C) 35; (D) 47; (E) 59.

3. Using the vertices of a cube as vertices, how many triangular pyramid can you form?

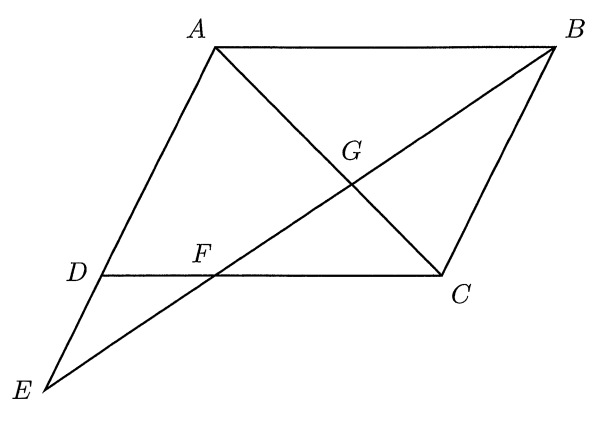
   (A) 54; (B) 58; (C) 60; (D) 64; (E) 70.

4. AB is a chord of a circle with centre O. CD is the diameter perpendicular to the chord AB, with AB closer to C than to D. Given that $\angle AOB = 90°$, then the quotient

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle AOD} = \_\_\_\_$$

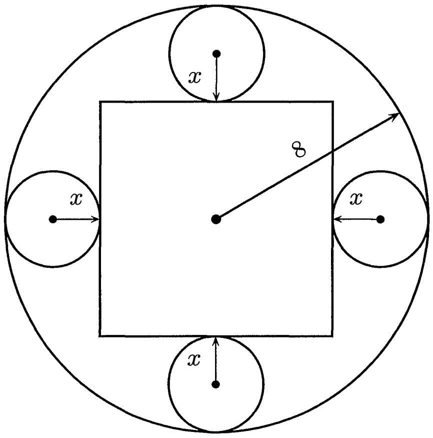
   (A) $\sqrt{2} - 1$; (B) $2 - \sqrt{2}$; (C) $\frac{\sqrt{2}}{2}$; (D) $\frac{1+\sqrt{2}}{2}$; (E) $\frac{1}{2}$.

5. The diagram below shows that $ABCD$ is a parallelogram and both AE and BE are straight lines. Let F and G be the intersections of BE with CD and AC respectively. Given that $BG = EF$, find the quotient DE AE.



   (A) $\frac{3-\sqrt{5}}{2}$; (B) $\frac{3-\sqrt{6}}{2}$; (C) $\frac{3-\sqrt{6}}{4}$; (D) $\frac{3-\sqrt{7}}{2}$; (E) $\frac{3-\sqrt{7}}{4}$.

6. Four circles each of radius x and a square are arranged within a circle of radius 8 as shown in the following figure.



What is the range of x?

(A) $0 < x < 4$; (B) $0 < x < 8(\sqrt{2}+1)$; (C) $4-2\sqrt{2} < x < 4$;

(D) $4-2\sqrt{2} < x < 8(\sqrt{2}-1)$; (E) $4-\sqrt{2} < x < 4(\sqrt{2}+1)$.

7. Adam has a triangular field ABC with $AB = 5$, $BC = 8$ and $CA = 11$. He intends to separate the field into two parts by building a straight fence from A to a point D on side BC such that AD bisects $\angle BAC$. Find the area of the part of the field ABD.

(A) $\frac{4\sqrt{21}}{11}$; (B) $\frac{4\sqrt{21}}{5}$; (C) $\frac{5\sqrt{21}}{11}$; (D) $\frac{5\sqrt{21}}{4}$; (E) None of the above.

8. For any real number x, let $\lfloor x \rfloor$ be the largest integer less than or equal to x and $\ x\ = x - \lfloor x \rfloor$. Let a and b be real numbers with $b \neq 0$ such that

$$a = b\left\lfloor \frac{a}{b} \right\rfloor - b\left\{\frac{a}{b}\right\}.$$

Which of the following statements is incorrect?

(A) If b is an integer then a is an integer;

(B) If a is a non-zero integer then b is an integer;

(C) If b is a rational number then a is a rational number;

(D) If a is a non-zero rational number then b is a rational number;

(E) If b is an even number then a is an even number.

9. Given that

$$y = \frac{\cancel{x} - \cancel{x}\,—}{x}$$

is an integer. Which of the following is incorrect?

(A) x can admit the value of any non-zero integer;

(B) x can be any positive number;

(C) x can be any negative number;

(D) y can take the value 2;

(E) y can take the value -2.

10. Suppose that A, B, C are three teachers working in three different schools X, Y, Z and specializing in three different subjects: Mathematics, Latin and Music. It is known that

(i) A does not teach Mathematics and B does not work in school Z;

(ii) The teacher in school Z teaches Music;

(iii) The teacher in school X does not teach Latin;

(iv) B does not teach Mathematics.

Which of the following statement is correct?

(A) B works in school X and C works in school Y;

(B) A teaches Latin and works in school Z;

(C) B teaches Latin and works in school Y;

(D) A teaches Music and C teaches Latin;

(E) None of the above.

Short Questions

11. Let a and b be real numbers such that $a > b$, $2^a + 2^b = 75$ and $2^{-a} + 2^{-b} = 12^{-1}$. Find the value of 2^{a-b}.

12. Find the sum of all positive integers x such that $\dfrac{x^3 - x + 120}{(x-1)(x+1)}$ is an integer.

13. Consider the equation

$$\sqrt{3x^2-8x+1}+\sqrt{9x^2-24x-8}=3.$$

It is known that the largest root of the equation is $-k$ times the smallest root. Find k.

14. Find the four-digit number $\overline{abcd}$ satisfying

$$2(\overline{abcd})+1000=\overline{dcba}.$$

(For example, if $a=1$, $b=2$, $c=3$ and $d=4$, then $\overline{abcd}=1234$.)

15. Suppose x and y are real numbers satisfying $x^2+y^2-22x-20y+221=0$. Find xy.

16. Let m and n be positive integers satisfying

$$mn^2+876=4mn+217n.$$

Find the sum of all possible values of m.

17. For any real number x, let $\lfloor x\rfloor$ denote the largest integer less than or equal to x. Find the value of $\lfloor x\rfloor$ of the smallest x satisfying $\lfloor x^2\rfloor-\lfloor x\rfloor^2=100$.

18. Suppose $x_1,x_2,\ldots,x_{49}$ are real numbers such that

$$x_1^2+2x_2^2+\cdots+49x_{49}^2=1.$$

Find the maximum value of $x_1+2x_2+\cdots+49x_{49}$.

19. Find the minimum value of

$$\sqrt{x^2+(20-y)^2}+\sqrt{y^2+(21-z)^2}+\sqrt{z^2+(20-w)^2}+\sqrt{w^2+(21-x)^2}.$$

20. Let A be a 4-digit integer. When both the first digit (left-most) and the third digit are increased by n, and the second digit and the fourth digit are decreased by n, the new number is n times A. Find the value of A.

21. Find the remainder when 1021^{1022} is divided by 1023.

22. Consider a list of six numbers. When the largest number is removed from the list, the average is decreased by 1. When the smallest number is removed, the average is increased by 1. When both the largest and the smallest numbers are removed, the average of the remaining four numbers is 20. Find the product of the largest and the smallest numbers.

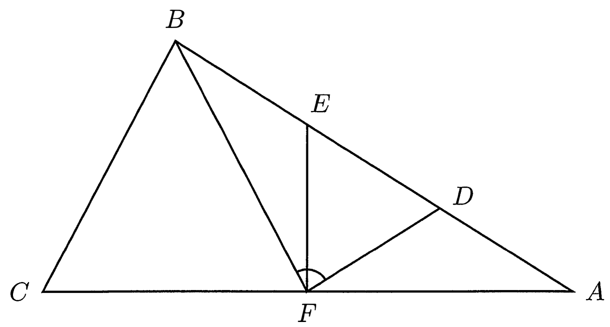
23. For each positive integer $n \geq 1$, we define the recursive relation given by

$$a_{n+1} = \frac{1}{1 + a_n}.$$

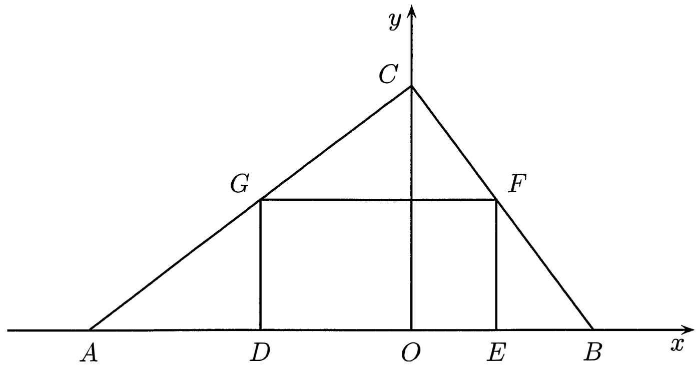
Suppose that $a_1 = a_{2012}$. Find the sum of the squares of all possible values of a_1.

24. A positive integer is called *friendly* if it is divisible by the sum of its digits. For example, 111 is friendly but 123 is not. Find the number of all two-digit friendly numbers.

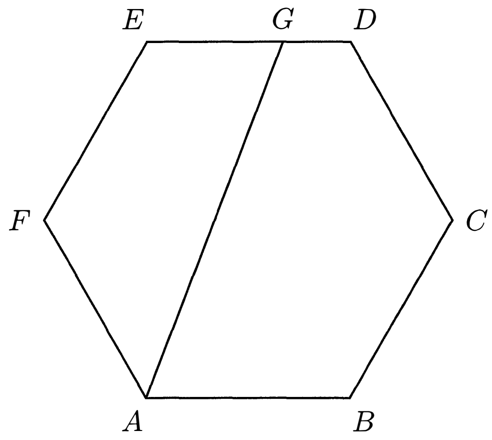
25. In the diagram below, D and E lie on the side AB, and F lies on the side AC such that $DA = DF = DE$, $BE = EF$ and $BF = BC$. It is given that $\angle ABC = 2\angle ACB$. Find x, where $\angle BFD = x°$.



26. In the diagram below, A and $B(20, 0)$ lie on the x-axis and $C(0, 30)$ lies on the y-axis such that $\angle ACB = 90°$. A rectangle $DEFG$ is inscribed in triangle ABC. Given that the area of triangle CGF is 351, calculate the area of the rectangle $DEFG$.



27. Let $ABCDEF$ be a regular hexagon. Let G be a point on ED such that $EG = 3GD$. If the area of $AGEF$ is 100, find the area of the hexagon $ABCDEF$.



28. Given a package containing 200 red marbles, 300 blue marbles and 400 green marbles. At each occasion, you are allowed to withdraw at most one red marble, at most two blue marbles and a total of at most five marbles out of the package. Find the minimal number of withdrawals required to withdraw all the marbles from the package.

29. 3 red marbles, 4 blue marbles and 5 green marbles are distributed to 12 students. Each student gets one and only one marble. In how many ways can the marbles be distributed so that Jamy and Jaren get the same colour and Jason gets a green marble?

30. A round cake is cut into n pieces with 3 cuts. Find the product of all possible values of n.

31. How many triples of non-negative integers (x, y, z) satisfying the equation

$$xyz + xy + yz + zx + x + y + z = 2012?$$

32. There are 2012 students in a secondary school. Every student writes a new year card. The cards are mixed up and randomly distributed to students. Suppose each student gets one and only one card. Find the expected number of students who get back their own cards.

33. Two players A and B play rock-paper-scissors continuously until player A wins 2 consecutive games. Suppose each player is equally likely to use each hand-sign in every game. What is the expected number of games they will play?

34. There are 2012 students standing in a circle; they are numbered $1, 2, \ldots, 2012$ clockwise. The counting starts from the first student (number 1) and proceeds around the circle clockwise. Alternate students will be eliminated from the circle in the following way: The first student stays in the circle while the second student leaves the circle. The third student stays while the fourth student leaves and so on. When the counting reaches number 2012, it goes back to number 1 and the elimination continues until the last student remains. What is the number of the last student?

35. There are k people and n chairs in a row, where $2 \leq k < n$. There is a couple among the k people. The number of ways in which all k people can be seated such that the couple is seated together is equal to the number of ways in which the $(k-2)$ people, without the couple present, can be seated. Find the smallest value of n.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

Junior Section (First Round)

Multiple Choice Questions

1. Answer: (D).

   Since $x^2 + 2bx + (b-1) = 0$, we have $\alpha\beta = b - 1$ and $\alpha + \beta = -2b$. Then

   $$\begin{aligned}(\alpha-\beta)^2 &= (\alpha+\beta)^2 - 4\alpha\beta = (-2b)^2 - 4(b-1)\\ &= 4(b^2 - b + 1) = 4[(b - 1\ \ 2)^2 + 3\ \ 4] \geq 3\end{aligned}$$

   The equality holds if and only if $b = 1\ \ 2$.

2. Answer: (E).

   Note that $n^{2012} + n^{2010} = n^{2010}(n^2+1)$.

   If $n = 2$, then $5\ \ (n^2+1)$ and $2\ \ n^{2010}$. If $n = 13$ or 47, then $10\ \ (n^2+1)$.

   If $n = 35$, then $2\ \ (n^2+1)$ and $5\ \ n^{2010}$. If $n = 59$, then $5 \nmid (59^2+1)$ and $5 \nmid n^{2010}$.

3. Answer: (B).

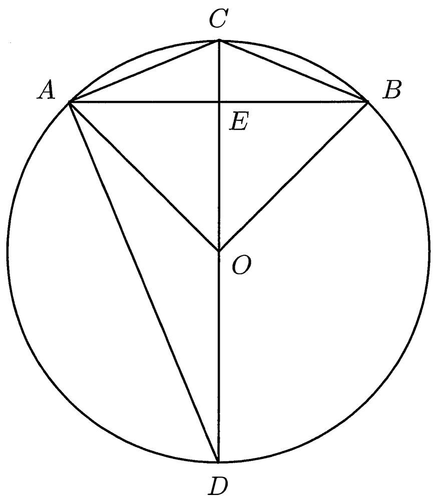
   There are 8 vertices in a cube. Any 4 vertices form a triangular pyramid unless they lie on the same plane.

   $$\binom{8}{4} - 6 - 6 = 58$$

4. Answer: (B).

   Let the radius of the circle be 1. Then

   $$\frac{S_{\triangle ABC}}{S_{\triangle AOD}} = \frac{AE \times CE}{S_{\triangle AOD}} = \frac{AE \times CE}{\frac{1}{2} \times AE \times OD} = \frac{2 \times CE}{OD} = \frac{2(1 - 1\ \ \sqrt{2})}{1} = 2 - \sqrt{2}$$



5. Answer: (A).

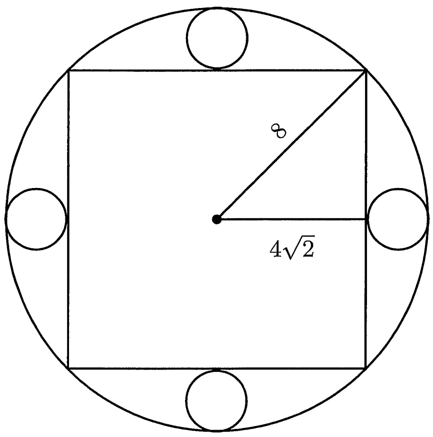
   Let $x = \dfrac{DE}{AE}$. Then $x = \dfrac{EF}{EB}$. Note that

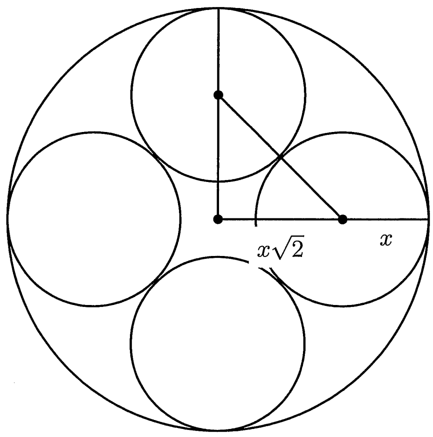
$$1 = \frac{DC}{AB} = \frac{DF}{AB} + \frac{FC}{AB} = \frac{EF}{EB} + \frac{FG}{GB} = \frac{EF}{EB} + \frac{EB - 2EF}{EF} = x + \left(\frac{1}{x} - 2\right)$$

   Then $x + \dfrac{1}{x} - 3 = 0$, and thus $x = \dfrac{3 - \sqrt{5}}{2}$ $(0 < x < 1)$.

6. Answer: (D).

   Consider the extreme cases:





   Then $x_{\min} = \dfrac{8 - 4\sqrt{2}}{2} = 4 - 2\sqrt{2}$ and $x_{\max} = \dfrac{8}{\sqrt{2}+1} = 8(\sqrt{2} - 1)$.

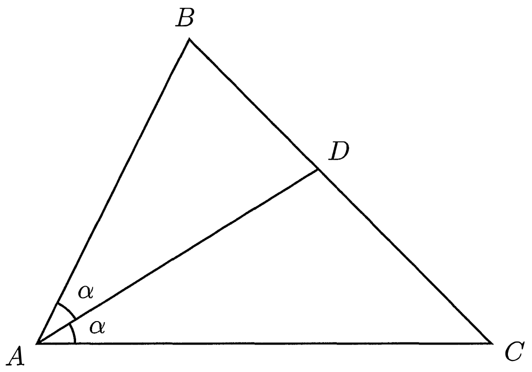
7. Answer: (D)

   By Heron's formula, $S_{\triangle ABC} = \sqrt{12(12-5)(12-8)(12-11)} = 4\sqrt{21}$.

$$\frac{S_{\triangle ABD}}{S_{\triangle ACD}} = \frac{\frac{1}{2} \times AB \times AD \times \sin\alpha}{\frac{1}{2} \times AC \times AD \times \sin\alpha} = \frac{AB}{AC} = \frac{5}{11}$$

   Then

$$S_{\triangle ABD} = \frac{5}{5+11} \times 4\sqrt{21} = \frac{5\sqrt{21}}{4}$$



8. Answer: (B).

   Note that $a = b \cdot \frac{a}{b} = b\left(\left\lfloor \frac{a}{b} \right\rfloor + \left\{\frac{a}{b}\right\}\right) = b\left\lfloor \frac{a}{b} \right\rfloor + b\left\{\frac{a}{b}\right\}$.

   It follows from $a = b\left\lfloor \frac{a}{b} \right\rfloor + b\left\{\frac{a}{b}\right\}$ that $b\left\lfloor \frac{a}{b} \right\rfloor = a$. Hence, $\frac{a}{b} = \left\lfloor \frac{a}{b} \right\rfloor$ is an integer.

   Obviously, b is not necessary an integer even if a is an integer.

9. Answer: (C).

   x can take any nonzero real number. If $x > 0$, then $y = \frac{x - x}{x} = \frac{x - x}{x} = \frac{0}{x} = 0$.

   If $x < 0$, then $y = \frac{x - x}{x} = \frac{x - (-x)}{x} = \frac{2x}{x} = \frac{-2x}{x} = -2$.

10. Answer: (C).

    The assignment is as follows:

    A: in Z, teaches Music; B: in Y, teaches Latin; C: in X, teaches Mathematics.

Short Questions

11. Answer: 4.

    $\frac{75}{12} = (2^a + 2^b)(2^{-a} + 2^{-b}) = 2 + 2^{a-b} + 2^{b-a}$. Then $2^{a-b} + 2^{b-a} = \frac{17}{4} = 4 + \frac{1}{4}$.

    Since $a > b$, we obtain $2^{a-b} = 4$.

12. Answer: 25.

    Note that $\frac{x^3 - x + 120}{(x-1)(x+1)} = x + \frac{120}{x^2 - 1}$. It is an integer if and only if $(x^2 - 1)$ –120.

    Then $x^2 - 1 = \pm 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 8\ \ 10\ \ 12\ \ 15\ \ 20\ \ 24\ \ 30\ \ 40\ \ 60\ \ 120$.

    Hence, $x = 0\ \ 2\ \ 3\ \ 4\ \ 5\ \ 11$; and $2 + 3 + 4 + 5 + 11 = 25$.

13. Answer: 9.

Let $y = \sqrt{3x^2 - 8x + 1}$. Then the equation becomes

$$y + \sqrt{3y^2 - 11} = 3.$$

Then $\sqrt{3y^2 - 11} = 3 - y$. Squaring both sides, we have $3y^2 - 11 = 9 - 6y + y^2$; that is, $y^2 + 3y - 10 = 0$. Then $y = 2$ or $y = -5$ (rejected because $y \geq 0$).

Solve $3x^2 - 8x + 1 = 2^2$. Then $x = 3$ and $x = -3^{-1}$. Hence, $k = 3/3^{-1} = 9$.

14. Answer: 2996.

Rewrite the equation as the following:

$$\begin{array}{ccccc} & a & b & c & d \\ & a & b & c & d \\ + & 1 & 0 & 0 & 0 \\ \hline & d & c & b & a \end{array}$$

Since a is the last digit of $2d$, a is even; since $2a + 1 \leq d \leq 9$, $a \leq 4$. So $a = 2$ or $a = 4$.

If $a = 4$, then $d \geq 2a + 1 = 9$ and thus $d = 9$; but then the last digit of $2d$ would be $8 \neq a$, a contradiction.

If $a = 2$, then $d \geq 2a + 1 = 5$ and the last digit of $2d$ is 2; so $d = 6$. The equation reduces to

$$\begin{array}{cccc} & & b & c \\ & & b & c \\ + & & & 1 \\ \hline & 1 & c & b \end{array}$$

There are 2 cases: either $2c + 1 = b$ and $2b = 10 + c$, which has no integer solution; or $2c + 1 = 10 + b$ and $2b + 1 = 10 + c$, which gives $b = c = 9$.

15. Answer: 110.

Complete the square: $(x - 11)^2 + (y - 10)^2 = 0$. Then $x = 11$ and $y = 10$; thus $xy = 110$.

16. Answer: 93.

Rearranging, we have

$$mn^2 - 217n = 4mn - 876 \Rightarrow n = \frac{4mn - 876}{mn - 217} = 4 - \frac{8}{mn - 217}.$$

Then $(mn - 217) \quad 8$. It follows that $mn - 217 = \pm 1, \pm 2, \pm 4, \pm 8$.

So $mn = 218, 216, 219, 215, 221, 213, 225, 209,$ and $n = -4, 12, 0, 8, 2, 6, 3, 5$ respectively.

Note that $m = \dfrac{mn}{n}$ is a positive integer. Then $m = \dfrac{216}{12} = 18$ or $m = \dfrac{225}{3} = 75$.

17. Answer: 50.

Write $x = \lfloor x \rfloor + \{x\}$. Then $100 \le (\lfloor x \rfloor + \{x\})^2 - \lfloor x \rfloor^2 = 2\lfloor x \rfloor\{x\} + \{x\}^2 < 2\lfloor x \rfloor + 1$.

So $\lfloor x \rfloor \ge 50$ and $x^2 \ge \lfloor x^2 \rfloor = 100 + 50^2 = 2600$. On the other hand, $x = \sqrt{2600}$ is a solution.

18. Answer: 35.

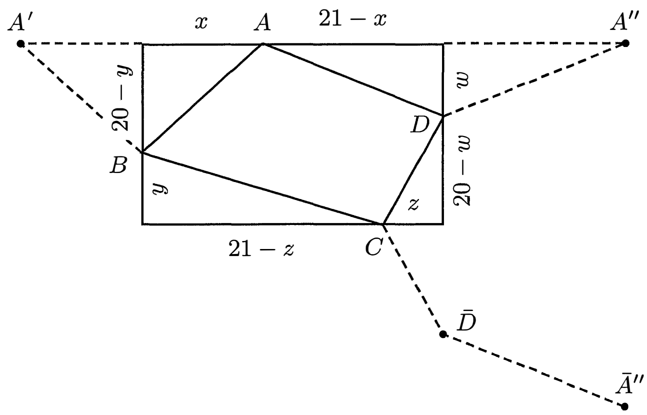
$$\begin{aligned}(x_1 + 2x_2 + \cdots + 49x_{49})^2 &= \left(1 \cdot x_1 + \sqrt{2} \cdot \sqrt{2}\, x_2 + \cdots + \sqrt{49} \cdot \sqrt{49}\, x_{49}\right)^2 \\ &\le (1 + 2 + \cdots + 49)\left(x_1^2 + 2x_2^2 + \cdots + 49x_{49}^2\right) \\ &= \frac{49 \times 50}{2} \times 1 = 35^2.\end{aligned}$$

The equality holds if and only if $x_1 = \cdots = x_{49} = \pm\dfrac{1}{35}$.

19. Answer: 58.

As shown below,

$$\begin{aligned}&\sqrt{x^2 + (20-y)^2} + \sqrt{y^2 + (21-z)^2} + \sqrt{z^2 + (20-w)^2} + \sqrt{w^2 + (21-x)^2} \\ &= AB + BC + CD + DA \\ &= A'B + BC + C\bar{D} + \bar{D}\bar{A}'' \\ &\le A'\bar{A}'' = \sqrt{42^2 + 40^2} = 58.\end{aligned}$$



20. Answer: 1818.

Let the 4-digit number be $A = \overline{abcd}$. Then

$$1000(a+n) + 100(b-n) + 10(c+n) + (d-n) = nA.$$

It gives $A + 909n = nA$; or equivalently, $(n-1)A = 909n$.

Note that $(n-1)$ and n are relatively prime and 101 is a prime number. We must have $(n-1) \mid 9$. So $n = 2$ or $n = 4$.

If $n = 4$, then $A = 1212$, which is impossible since $b < n$. So $n = 2$ and $A = 909 \times 2 = 1818$.

21. Answer: 4.

Note that $1024 = 2^{10} \equiv 1 \pmod{1023}$. Then

$$1021^{1022} \equiv (-2)^{1022} \equiv 2^{1022} \equiv 2^{10\times 102+2} \equiv 1024^{102} \times 2^2 \equiv 1^{102} \times 2^2 \equiv 4 \pmod{1023}$$

22. Answer: 375.

Let m and M be the smallest and the largest numbers. Then

$$\frac{m+80}{5} + 1 = \frac{M+80}{5} - 1 = \frac{m+M+80}{6}$$

Solving the system, $m = 15$ and $M = 25$. Then $mM = 375$.

23. Answer: 3.

Let $a_1 = a$. Then $a_2 = \dfrac{1}{1+a}$, $a_3 = \dfrac{1+a}{2+a}$, $a_4 = \dfrac{2+a}{3+2a}$, $a_5 = \dfrac{3+2a}{5+3a}$, ... In general,

$$a_n = \frac{F_n + F_{n-1}a}{F_{n+1} + F_n a}$$

where $F_1 = 0$, $F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$.

If $a_{2012} = \dfrac{F_{2012} + F_{2011}a}{F_{2013} + F_{2012}a} = a$, then $(a^2 + a - 1)F_{2012} = 0$.

Since $F_{2012} > 0$, we have $a^2 + a - 1 = 0$. Let α and β be the roots of $a^2 + a - 1 = 0$. Then

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (-1)^2 - 2(-1) = 3$$

24. Answer: 23.

Since $\dfrac{10a+b}{a+b} = \dfrac{9a}{a+b} + 1$, we have $(a+b) \quad 9a$.

Case 1: If $3 \nmid (a+b)$, then $(a+b) \quad a$, and thus $b = 0$. We have 10 20 40 50 70 80.

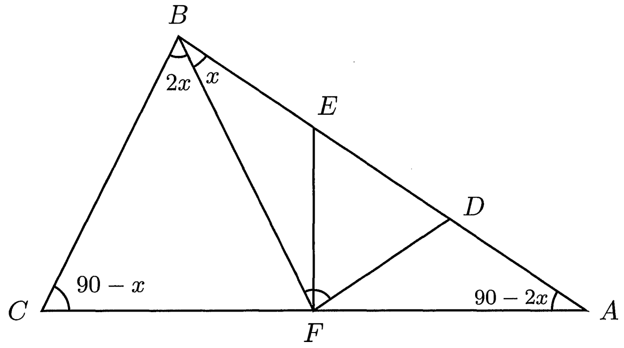
Case 2: If $3 \quad (a+b)$ but $9 \nmid (a+b)$, then $(a+b) \quad 3a$ and $1 \leq \dfrac{3a}{a+b} \leq 3$. If $\dfrac{3a}{a+b} = 1$, then $2a = b$, we have 12 24 48. If $\dfrac{3a}{a+b} = 2$, then $a = 2b$, we have 21 42 84. If $\dfrac{3a}{a+b} = 3$, then $b = 0$, we have 30 60.

Case 3: If $9 \quad (a+b)$, then $a+b = 9$ or 18. If $a+b = 18$, then $a = b = 9$, which is impossible. If $a+b = 9$, then we have 9 friendly numbers 18 27 36 45 54 63 72 81 90.

Therefore, in total there are 23 2-digit friendly numbers.

25. Answer: 108.

Since $DA = DE = DF$, $\angle EFA = 90^\circ$. Let $\angle EBF = \angle EFB = x^\circ$. Then $\angle BCF = \angle BFC = 90^\circ - x^\circ$ and $\angle CBF = 2x^\circ$, $\angle BAC = 90^\circ - 2x^\circ$.



It is given that $3x = 2(90 - x)$. Then $x = 36$. So $x = 180 - (90 - x) - (90 - 2x) = 3x = 108$.

26. Answer: 468.

Note that $AO = \dfrac{30^2}{20} = 45$. Then the area of $\triangle ABC$ is $\dfrac{(20 + 45) \times 30}{2} = 975$.

Let the height of $\triangle CGF$ be h. Then

$$\left(\frac{h}{30}\right)^2 = \frac{351}{975} = \left(\frac{3}{5}\right)^2 \Rightarrow \frac{h}{30 - h} = \frac{3}{2}$$

Note that the rectangle $DEFG$ has the same base as $\triangle CGF$. Then its area is

$$351 \times \frac{2}{3} \times 2 = 468$$

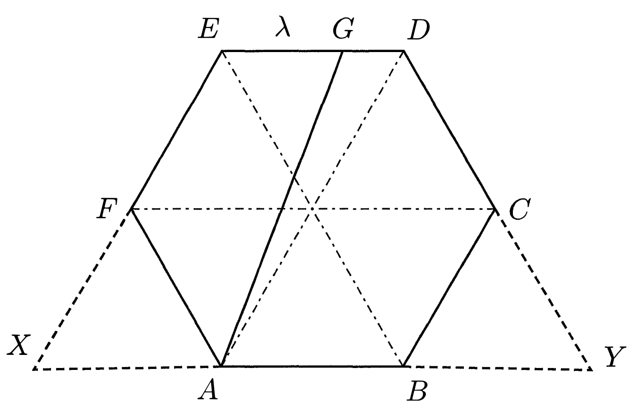
27. Answer: 240.

We may assume that $DE = 1$ and denote $\lambda = EG = 3\ 4$.

$\dfrac{EGAX}{DEXY} = \dfrac{1 + \lambda}{4}$ implies $\dfrac{AGEF}{DEXY} = \dfrac{1 + \lambda}{4} - \dfrac{1}{8} = \dfrac{1 + 2\lambda}{8}$.

Note that $\dfrac{ABCDEF}{DEXY} = \dfrac{6}{8}$. Then $\dfrac{AGEF}{ABCDEF} = \dfrac{1 + 2\lambda}{6} = \dfrac{5}{12}$.

If $AGEF = 100$, then $ABCDEF = 100 \times \dfrac{12}{5} = 240$.



28. Answer: 200.

Since at most one red marble can be withdrawn each time, it requires at least 200 withdrawals.

On the other hand, 200 is possible. For example,

$$150 \cdot (1,2,2) + 20 \cdot (1,0,4) + 10 \cdot (1,0,2) + 20 \cdot (1,0,0) = (200,300,400).$$

29. Answer: 3150.

Case 1: Jamy and Jaren both take red marbles. So 1 red, 4 blue and 4 green marbles are distributed to 9 students:

$$\binom{9}{1} \times \binom{8}{4} = 630.$$

Case 2: Jamy and Jaren both take blue marbles. So 3 red, 2 blue and 4 green marbles are distributed to 9 students:

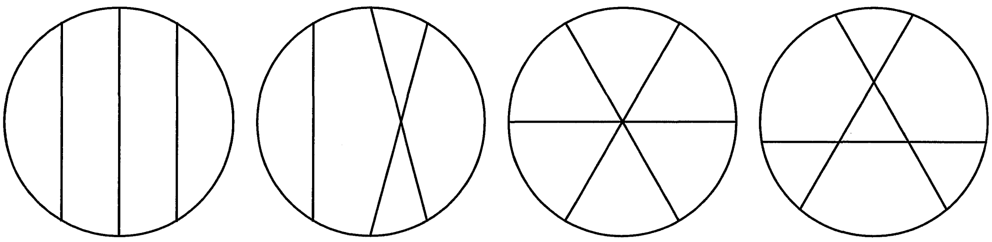
$$\binom{9}{3} \times \binom{6}{2} = 1260.$$

Case 3: Jamy and Jaren both take green marbles. So 3 red, 4 blue and 2 green marbles are distributed to 9 students. The number is the same as case 2.

$$630 + 1260 + 1260 = 3150.$$

30. Answer: 840.

With three cuts, a round cake can be cut into at least $1+3=4$ pieces, and at most $1+1+2+3=7$ pieces. Moreover, $n = 4, 5, 6, 7$ are all possible. $4 \times 5 \times 6 \times 7 = 840$.



31. Answer: 27.

$(x+1)(y+1)(z+1) = 2013 = 3 \times 11 \times 61$.

If all x, y, z are positive, there are $3! = 6$ solutions.

If exactly one of x, y, z is 0, there are $3 \times 6 = 18$ solutions.

If exactly two of x, y, z are 0, there are 3 solutions.

$6 + 18 + 3 = 27$.

32. Answer: 1.

For each student, the probability that he gets back his card is $\dfrac{1}{2012}$. Then the expectation of the whole class is $2012 \times \dfrac{1}{2012} = 1$.

33. Answer: 12.

Let E be the expectation. If A does not win, the probability is $2/3$ and the game restarts. If A wins and then does not win, the probability is $(1/3)(2/3)$ and the game restarts. The probability that A wins two consecutive games is $(1/3)(1/3)$. Then

$$E = \frac{2}{3} \times (E+1) + \frac{2}{9} \times (E+2) + \frac{1}{9} \times 2.$$

Solving the equation, we get $E = 12$.

34. Answer: 1976.

If there are $1024 = 2^{10}$ students, then the 1024^{th} student is the last one leaving the circle. Suppose $2012 - 1024 = 988$ students have left. Among the remaining 1024 students, the last student is $(2 \times 988 - 1) + 1 = 1976$.

35. Answer: 12.

$(n-1) \times 2 \times \binom{n-2}{k-2} \times (k-2)! = \binom{n}{k-2} \times (k-2)!$. Then $2 = \dfrac{n}{(n-k+1)(n-k+2)}$.

That is, $2n^2 - (4k-6)n + (2k^2 - 6k + 4 - n) = 0$. We can solve

$$n = k + \frac{\sqrt{8k-7} - 5}{4} \qquad (n > k).$$

Note that the square of any odd number has the form $8k-7$. Choose k so that $\sqrt{8k-7} - 5 = 4$, i.e., $k = 11$. Then $n = 12$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

(Junior Section, Round 2)

Saturday, 23 June 2012 **0930-1230**

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1. Let O be the centre of a parallelogram $ABCD$ and P be any point in the plane. Let M, N be the midpoints of AP, BP, respectively and Q be the intersection of MC and ND. Prove that O, P and Q are collinear.

2. Does there exist an integer A such that each of the ten digits $0, 1, \ldots, 9$ appears exactly once as a digit in exactly one of the numbers A, A^2, A^3.

3. In $\triangle ABC$, the external bisectors of $\angle A$ and $\angle B$ meet at a point D. Prove that the circumcentre of $\triangle ABD$ and the points C, D lie on the same straight line.

4. Determine the values of the positive integer n for which the following system of equations has a solution in positive integers $x_1, x_2, \ldots, x_n$. Find all solutions for each such n.

$$x_1 + x_2 + \cdots + x_n = 16 \tag{1}$$

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1 \tag{2}$$

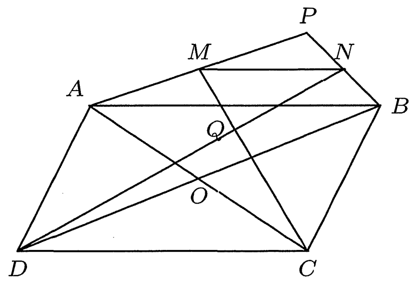
5. Suppose $S = \ a_1, a_2, \ldots, a_{15}\ $ is a set of 15 distinct positive integers chosen from $\ 2, 3, \ldots, 2012\ $ such that every two of them are coprime. Prove that S contains a prime number. (Note" Two positive integers m, n are coprime if their only common factor is 1.)

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

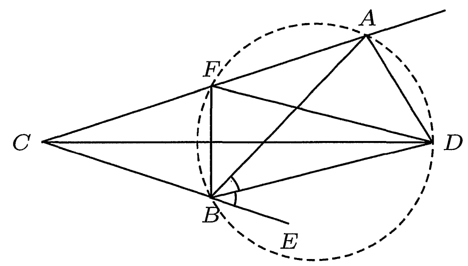
(Junior Section, Round 2 solutions)

1. Since $MN \parallel AB \parallel CD$, we have $\triangle MQN \sim \triangle CDQ$. Hence $MN = AB/2 = CD/2$. Thus $QM = CQ/2$. In $\triangle ACP$, CM is a median and Q divides CM in the ratio 1:2. Thus Q is the centroid. Hence the median PO passes through Q.



2. Since the total number of digits in A, A^2 and A^3 is 10, the total number digits in $32, 32^2, 32^3$ is 11 and the total number of digits in $20, 20^2, 20^3$ is 9, any solution A must satisfy $21 \leq A \leq 31$. Since the unit digits of A, A^2, A^3 are distinct, the unit digit of A can only be $2, 3, 7, 8$. Thus the only possible values of A are $22, 23, 27, 28$. None of them has the desired property. Thus no such number exists.

3. Note that CD bisects $\angle C$. If $CA = CB$, then CD is the perpendicular bisector of AB. Thus the circumcentre of $\triangle ABD$ is on CD.



If $CA \neq CB$, we may assume that $CA > CB$. Let E be a point on CB extended and F be the point on CA so that $CF = CB$. Then, since CD is the perpendicular bisector of

BF, we have $\angle AFD = \angle DBE = \angle DBA$. Thus $AFBD$ is a cyclic quadrilateral, i.e., F is on the circumcircle of $\triangle ABD$. The circumcentre lies on the perpendicular bisector of BF which is CD.

4. Without loss of generality, we may assume that $x_1 \le x_2 \le \cdots \le x_n$. If $x_1 = 1$, then from (2), $n = 1$ and (1) cannot be satisfied. Thus $x_1 \ge 2$. If $x_2 = 2$, then $n = 2$ and again (1) cannot be satisfied. Thus $x_2 \ge 3$. Similarly, $x_3 \ge 4$. Thus $x_4 + \cdots + x_n \le 7$ with $x_4 \ge 4$. Thus $n \le 4$.

(i) $n = 1$: No solution.

(ii) $n = 2$: The only solution of $\frac{1}{x_1} + \frac{1}{x_2} = 1$ is $x_1 = x_2 = 2$ which doesn't satisfy (1). Thus there is no solution.

(iii) $n = 3$: The only solutions of $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 1$ are $(x_1, x_2, x_3) = (2,3,6), (2,4,4)$ and $(3,3,3)$. They all do not satisfy (1).

(iv) $n = 4$: According to the discussion in the first paragraph, the solutions of $x_1 + \cdots + x_4 = 16$ are

$$\begin{aligned}(x_1, x_2, x_3, x_4) = {} & (2,3,4,7), (2,3,5,6), (2,4,4,6), (2,4,5,5), \\ & (3,3,4,6), (3,3,5,5), (3,4,4,5), (4,4,4,4).\end{aligned}$$

Only the last one satisfy (2).

Thus the system of equations has a solution only when $n = 4$ and for this n, the only solution is $x_1 = x_2 = x_3 = x_4 = 4$.

5. Suppose, on the contrary, that S contains no primes. For each i, let p_i be the smallest prime divisor of a_i. Then $p_1, p_2, \ldots, p_{15}$ are distinct since the numbers in S are pairwise coprime. The first 15 primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. If p_j is the largest among $p_1, p_2, \ldots, p_{15}$, then $p_j \ge 47$ and $a_j \ge 47^2 = 47^2 = 2309 > 2012$, a contradiction. Thus S must contain a prime number.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

Senior Section (First Round)

Tuesday, 29 May 2012 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x. *For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Suppose α and β are real numbers that satisfy the equation

$$x^2 + \left(2\sqrt{\sqrt{2}+1}\right)x + \sqrt{\sqrt{2}+1} = 0$$

Find the value of $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3}$.

(A) $3\sqrt{\sqrt{2}+1}\,(\sqrt{2}-1) - 8$ (B) $8 - 6\sqrt{\sqrt{2}+1}\,(\sqrt{2}-1)$

(C) $3\sqrt{\sqrt{2}+1}\,(\sqrt{2}-1) + 8$ (D) $6\sqrt{\sqrt{2}+1}\,(\sqrt{2}-1) - 8$

(E) None of the above

2. Find the value of

$$\frac{2011^2 \times 2012 - 2013}{2012!} + \frac{2013^2 \times 2014 - 2015}{2014!}$$

(A) $\frac{1}{2009!} + \frac{1}{2010!} + \frac{1}{2011!} + \frac{1}{2012!}$ (B) $\frac{1}{2009!} + \frac{1}{2010!} - \frac{1}{2011!} - \frac{1}{2012!}$

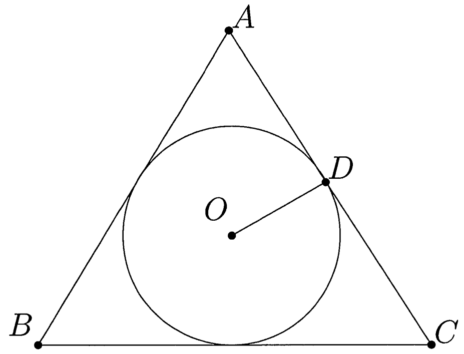
(C) $\frac{1}{2009!} + \frac{1}{2010!} - \frac{1}{2012!} - \frac{1}{2013!}$ (D) $\frac{1}{2009!} + \frac{1}{2010!} + \frac{1}{2013!} + \frac{1}{2014!}$

(E) $\frac{1}{2009!} + \frac{1}{2010!} - \frac{1}{2013!} - \frac{1}{2014!}$

3. The increasing sequence $T =$ 2 3 5 6 7 8 10 11 consists of all positive integers which are not perfect squares. What is the 2012th term of T?

(A) 2055 (B) 2056 (C) 2057 (D) 2058 (E) 2059

4. Let O be the center of the inscribed circle of triangle $\triangle ABC$ and D be the point on AC with $OD \perp AC$. If $AB = 10$ $AC = 9$ $BC = 11$, find CD.



(A) 4 (B) 4.5 (C) 5 (D) 5.5 (E) 6

5. Find the value of

$$\frac{\cos^4 75^\circ + \sin^4 75^\circ + 3\sin^2 75^\circ \cos^2 75^\circ}{\cos^6 75^\circ + \sin^6 75^\circ + 4\sin^2 75^\circ \cos^2 75^\circ}.$$

(A) $\frac{\sqrt{2}}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) 1 (E) $\cos 75^\circ + \sin 75^\circ$

6. If the roots of the equation $x^2 + 3x - 1 = 0$ are also the roots of the equation $x^4 + ax^2 + bx + c = 0$, find the value of $a + b + 4c$.

(A) -13 (B) -7 (C) 5 (D) 7 (E) 11

7. Find the sum of the **digits** of all numbers in the sequence $1, 2, 3, 4, \ldots, 1000$.

(A) 4501 (B) 12195 (C) 13501 (D) 499500 (E) None of the above

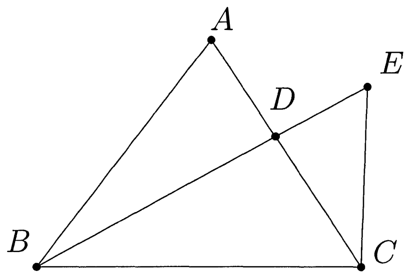
8. Find the number of real solutions to the equation

$$\frac{x}{100} = \sin x,$$

where x is measured in radians.

(A) 30 (B) 32 (C) 62 (D) 63 (E) 64

9. In the triangle $\triangle ABC$, $AB = AC$, $\angle ABC = 40^\circ$ and the point D is on AC such that BD is the angle bisector of $\angle ABC$. If BD is extended to the point E such that $DE = AD$, find $\angle ECA$.



(A) 20° (B) 30° (C) 40° (D) 45° (E) 50°

10. Let m and n be positive integers such that $m > n$. If the last three digits of 2012^m and 2012^n are identical, find the smallest possible value of $m + n$.

(A) 98 (B) 100 (C) 102 (D) 104 (E) None of the above

Short Questions

11. Let a b c and d be four distinct positive real numbers that satisfy the equations

$$(a^{2012} - c^{2012})(a^{2012} - d^{2012}) = 2011$$

and

$$(b^{2012} - c^{2012})(b^{2012} - d^{2012}) = 2011$$

Find the value of $(cd)^{2012} - (ab)^{2012}$.

12. Determine the total number of pairs of integers x and y that satisfy the equation

$$\frac{1}{y} - \frac{1}{y+2} = \frac{1}{3 \cdot 2^x}$$

13. Given a set $S = \quad 1 \; 2 \qquad 10 \;$, a collection $\mathcal{F}$ of subsets of S is said to be *intersecting* if for any two subsets A and B in $\mathcal{F}$, we have $A \cap B \neq \emptyset$. What is the maximum size of $\mathcal{F}$?

14. The set M contains all the integral values of m such that the polynomial

$$2(m-1)x^2 - (m^2 - m + 12)x + 6m$$

has either one repeated or two distinct integral roots. Find the number of elements of M.

15. Find the minimum value of

$$\left| \sin x + \cos x + \frac{\cos x - \sin x}{\cos 2x} \right|$$

16. Find the number of ways to arrange the letters A, A, B, B, C, C, D and E in a line, such that there are no consecutive identical letters.

17. Suppose $x = 3^{\sqrt{2+\log_3 x}}$ is an integer. Determine the value of x.

18. Let $f(x)$ be the polynomial $(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5)$ where a_1, a_2, a_3, a_4 and a_5 are distinct integers. Given that $f(104) = 2012$, evaluate $a_1+a_2+a_3+a_4+a_5$.

19. Suppose x, y, z and λ are positive real numbers such that

$$\begin{aligned} yz &= 6\lambda x \\ xz &= 6\lambda y \\ xy &= 6\lambda z \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

Find the value of $(xyz\lambda)^{-1}$.

20. Find the least value of the expression $(x+y)(y+z)$, given that x, y, z are positive real numbers satisfying the equation

$$xyz(x+y+z) = 1.$$

21. For each real number x, let $f(x)$ be the minimum of the numbers $4x+1, x+2$ and $-2x+4$. Determine the maximum value of $6f(x)+2012$.

22. Find the number of pairs (A, B) of distinct subsets of $\ 1,2,3,4,5,6\ $ such that A is a proper subset of B. Note that A can be an empty set.

23. Find the sum of all the integral values of x that satisfy

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1.$$

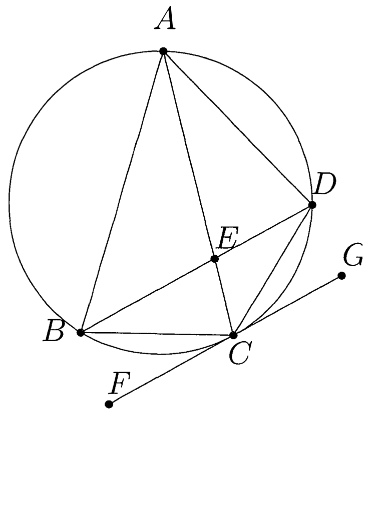
24. Given that

$$S = \left|\sqrt{x^2+4x+5}-\sqrt{x^2+2x+5}\right|,$$

for real values of x, find the maximum value of S^4.

25. Three integers are selected from the set $S = \ 1, 2, 3, \ldots, 19, 20\ $. Find the number of selections where the sum of the three integers is divisible by 3.

26. In the diagram below, $ABCD$ is a cyclic quadrilateral with $AB = AC$. The line FG is tangent to the circle at the point C, and is parallel to BD. If $AB = 6$ and $BC = 4$, find the value of $3AE$.



27. Two Wei Qi teams, A and B, each comprising of 7 members, take on each other in a competition. The players on each team are fielded in a fixed sequence. The first game is played by the first player of each team. The losing player is eliminated while the winning player stays on to play with the next player of the opposing team. This continues until one team is completely eliminated and the surviving team emerges as the final winner – thus, yielding a possible gaming outcome. Find the total number of possible gaming outcomes.

28. Given that $\mathbf{m} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$ and $\mathbf{n} = (\sqrt{2} - \sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the usual unit vectors along the x-axis and the y-axis respectively, and $\theta \in (\pi \ \ 2\pi)$. If the length or magnitude of the vector $\mathbf{m} + \mathbf{n}$ is given by $\ \mathbf{m} + \mathbf{n} \ = \frac{8\sqrt{2}}{5}$, find the value of $5\cos\left(\frac{\theta}{2} + \frac{\pi}{8}\right) + 5$.

29. Given that the real numbers x, y and z satisfies the condition $x + y + z = 3$, find the maximum value of $f(x \ y \ z) = \sqrt{2x+13} + \sqrt[3]{3y+5} + \sqrt[4]{8z+12}$.

30. Let $P(x)$ be a polynomial of degree 34 such that $P(k) = k \ \ (k+1)$ for all integers $k = 0 \ 1 \ 2 \qquad 34$. Evaluate $42840 \times P(35)$.

31. Given that α is an acute angle satisfying

$$\sqrt{369 - 360\cos\alpha} + \sqrt{544 - 480\sin\alpha} - 25 = 0$$

find the value of $40\tan\alpha$.

32. Given that $a \ b \ c \ d \ e$ are real numbers such that

$$a + b + c + d + e = 8$$

and

$$a^2 + b^2 + c^2 + d^2 + e^2 = 16$$

Determine the maximum value of $\lfloor e \rfloor$.

33. Let L denote the minimum value of the quotient of a 3-digit number formed by three distinct digits divided by the sum of its digits. Determine $\lfloor 10L \rfloor$.

34. Find the last 2 digits of

$$x = 19^{17^{15^{\cdot^{\cdot^{\cdot^{3^1}}}}}}$$

35. Let $f(n)$ be the integer nearest to $\sqrt{n}$. Find the value of

$$\sum_{n=1}^{\infty} \frac{\left(\frac{3}{2}\right)^{f(n)} + \left(\frac{3}{2}\right)^{-f(n)}}{\left(\frac{3}{2}\right)^n}$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

Senior Section (First Round Solutions)

Multiple Choice Questions

1. Answer: (D)

   Note that $\alpha+\beta=-\left(2\sqrt{\sqrt{2}+1}\right)$ and $\alpha\beta=\sqrt{\sqrt{2}+1}$.

$$
\begin{aligned}
\frac{1}{\alpha^3}+\frac{1}{\beta^3} &= \frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}\\
&= \frac{6-8\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}+1}}\\
&= \frac{6(\sqrt{\sqrt{2}+1})-8(\sqrt{2}+1)}{\sqrt{2}+1}\\
&= 6\sqrt{\sqrt{2}+1}\left(\sqrt{2}-1\right)-8
\end{aligned}
$$

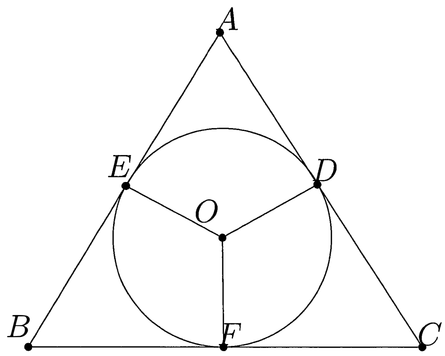
2. Answer: (E)

$$
\begin{aligned}
&\frac{2011^2\times 2012-2013}{2012!}+\frac{2013^2\times 2014-2015}{2014!}\\
&=\frac{2011}{2010!}-\frac{2013}{2012!}+\frac{2013}{2012!}-\frac{2015}{2014!}\\
&=\left(\frac{2010}{2010!}+\frac{1}{2010!}-\frac{2012}{2012!}-\frac{1}{2012!}\right)+\left(\frac{2012}{2012!}+\frac{1}{2012!}-\frac{2014}{2014!}-\frac{1}{2014!}\right)\\
&=\left(\frac{1}{2009!}+\frac{1}{2010!}-\frac{1}{2011!}-\frac{1}{2012!}\right)+\left(\frac{1}{2011!}+\frac{1}{2012!}-\frac{1}{2013!}-\frac{1}{2014!}\right)\\
&=\frac{1}{2009!}+\frac{1}{2010!}-\frac{1}{2013!}-\frac{1}{2014!}
\end{aligned}
$$

3. Answer: (C)

   Note that $44^2 = 1936$, $45^2 = 2025$ and $46^2 = 2116$. So $2, 3, \ldots, 2012$ has at most $2012 - 44$ terms. For the 2012th term, we need to add the 44 numbers from 2013 to 2056. But in doing so, we are counting $45^2 = 2025$, so the 2012th term should be $2012 + 44 + 1 = 2057$.

4. Answer: (C)



   AC is tangent to the circle at D, by constructing E and F as shown, we have $CD = CF$, $AD = AE$ and $BE = BF$. Solving for the unknowns give $CD = 5$.

5. Answer: (D)

$$
\begin{aligned}
&\cos^6 x + \sin^6 x + 4\sin^2 x \cos^2 x \\
&= (\cos^2 x + \sin^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x) + 4\sin^2 x \cos^2 x \\
&= (\sin^4 x + \cos^4 x + 3\sin^2 x \cos^2 x).
\end{aligned}
$$

6. Answer: (B)

   We have the factorization

$$(x^2 + 3x - 1)(x^2 + mx - c) = x^4 + ax^2 + bx + c.$$

   Comparing coefficients give $3 + m = 0$, $-1 - c + 3m = a$ and $-3c - m = b$. We can solve these equations to obtain $a + b + 4c = -7$.

7. Answer: (C)

   Among all numbers with 3 or less digits, each i, $i = 0, 1, 2, \cdots, 9$, appears exactly 300 times. Thus the sum of the digits of all the numbers in the sequence $1, 2, 3, 4, \cdots, 999$ is

$$300(1 + 2 + \cdots + 9) = 13500,$$

   and so the answer is 13501.

8. Answer: (D)

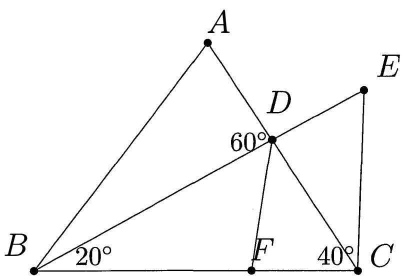
   Since $-1 \leq \sin x \leq 1$, we have $-100 \leq x \leq 100$. We also observe that $31\pi < 100 < 32\pi$.

   For each integer k with $1 \leq k \leq 16$, $\frac{x}{100} = \sin x$ has exactly two solutions in $[2(k-1)\pi \quad (2k-1)\pi]$, but it has no solutions in $((2k-1)\pi \quad 2k\pi)$. Thus this equation has exactly 32 non-negative real solutions, i.e. $x = 0$ and exactly 31 positive real solutions. Then it also has exactly 31 negative real solutions, giving a total of 63.

9. Answer: (C)

   Construct a point F on BC such that $BF = BA$. Since $\angle ABD = \angle FBD$, $\triangle ABD$ is congruent to $\triangle FBD$.

   Thus $DF = DA = DE$ and $\angle FDB = \angle ADB = 60°$. We also have $\angle EDC = \angle ADB = 60°$, which implies that $\angle FDC = 60°$ and $\triangle CFD$ is congruent to $\triangle CED$. In conclusion, $\angle ECD = \angle FCD = 40°$.



10. Answer: (D)

    We want to solve $2012^m \equiv 2012^n \pmod{1000}$ which is equivalent to

$$1000 \quad 12^n(12^{m-n} - 1)$$

    Since $(12^{m-n} - 1)$ is odd, we must have $8 \quad 12^n$ so $n \geq 2$. It remains to check that $125 \quad 12^{m-n} - 1$ i.e. $12^{m-n} \equiv 1 \pmod{125}$. Let φ be Euler's phi function. As $\varphi(125) = 125 - 25 = 100$, by Euler's theorem, we know that the smallest $m - n$ must be a factor of 100. By checking all possible factors, we can conclude $m - n = 100$ and so the smallest possible value for $m + n$ is 104 since $n \geq 2$.

Short Questions

11. Answer: 2011

Let $A = a^{2012}$, $B = b^{2012}$, $C = c^{2012}$ and $D = d^{2012}$. Then A and B are distinct roots of the equation $(x - C)(x - D) = 2011$. Thus the product of roots, $AB = CD - 2011$ and $CD - AB = 2011$.

12. Answer: 6

Note that x and y must satisfy

$$2^{x+1} \cdot 3 = y(y+2).$$

We first assume $x \geq 0$, which means both y and $y+2$ are even integers. Either $3 \quad y$ or $3 \quad y+2$. In the first case, assuming $y > 0$, we have $y = 3 \cdot 2^k$ and $y + 2 = 2(3 \cdot 2^{k-1} + 1) = 2^{x+1-k}$. The only way for this equation to hold is $k = 1$ and $x = 3$. So $(x, y, y+2) = (3, 6, 8)$.

In the case $3 \quad y+2$, assuming $y > 0$, we have $y + 2 = 3 \cdot 2^k$ and $y = 2(3 \cdot 2^{k-1} - 1) = 2^{x+1-k}$. Now the only possibility is $k = 1$ and $x = 2$, so $(x, y, y+2) = (2, 4, 6)$.

In the two previous cases, we could also have both y and $y+2$ to be negative, giving $(x, y, y+2) = (3, -8, -6)$ or $(2, -6, -4)$.

Finally, we consider $x < 0$ so $3 = 2^{-x-1} y(y+2)$. In this case we can only have $x = -1$ and $(x, y, y+2) = (-1, 1, 3)$ or $(-1, -3, -1)$.

Hence possible (x, y) pairs are $(3, 6)$, $(3, -8)$, $(2, 4)$, $(2, -6)$, $(-1, 1)$ and $(-1, -3)$.

13. Answer: 512

If $A \in \mathcal{F}$ then the complement $S \cap A \notin \mathcal{F}$. So at most half of all the subsets of S can belong to $\mathcal{F}$, that is

$$\mathcal{F} \leq \frac{2^{10}}{2} = 512.$$

Equality holds because we can take $\mathcal{F}$ to be all subsets of S containing 1.

14. Answer: 4

If $m = 1$, the polynomial reduces to $-12x + 6 = 0$ which has no integral roots.

For $m \neq 1$, the polynomial factorizes as $((2x - m)((m-1)x - 6)$, with roots $x = \frac{m}{2}$ and $x = \frac{6}{m-1}$. For integral roots, m must be even and $m - 1$ must divide 6. The only possible values are $m = -2, 0, 2$ and 4. So M has 4 elements.

15. Answer: 2

Note that $\cos 2x = \cos^2 x - \sin^2 x$. So

$$\left|\sin x + \cos x + \frac{\cos x - \sin x}{\cos 2x}\right| = \left|\sin x + \cos x + \frac{1}{\sin x + \cos x}\right|.$$

Set $w = \sin x + \cos x$ and minimize $\left|w + \frac{1}{w}\right|$.

By AM-GM inequality, if w is positive then the minimum of $w + \frac{1}{w}$ is 2; if w is negative, then the maximum of $w + \frac{1}{w}$ is -2. Therefore, the minimum of $\left|w + \frac{1}{w}\right|$ is 2.

16. Answer: 2220

We shall use the Principle of Inclusion and Exclusion. There are $\frac{8!}{2!2!2!}$ ways to arrange the letters without restriction. There are $\frac{7!}{2!2!}$ ways to arrange the letters such that both the As occur consecutively. (Note that this is the same number if we use B or C instead of A.)

There are $\frac{6!}{2!}$ ways to arrange the letters such that both As and Bs are consecutive. (Again this number is the same for other possible pairs.) Finally there are 5! ways to arrange the letters such that As, Bs and Cs occur consecutively.

For there to be no consecutive identical letters, total number of ways is

$$\frac{8!}{2!2!2!} - 3 \times \frac{7!}{2!2!} + 3 \times \frac{6!}{2!} - 5! = 2220$$

17. Answer: 9

Taking logarithm, we get $\log_3 x = \sqrt{2 + \log_3 x}$ Let $y = \log_3 x$. The only possible solution for $y = \sqrt{2+y}$ is 2. Therefore $x = 3^2 = 9$.

18. Answer: 17

The prime factorization of 2012 is $2^2 \cdot 503$. Let $b = 104$. If a_i are distinct, so are $b - a_i$, i.e. $(b - a_1)$, $(b - a_2)$, $(b - a_3)$, $(b - a_4)$ and $(b - a_5)$ must be exactly the integers $1\ -1\ 2\ -2\ 503$. Summing up, we have

$$5(104) - (a_1 + a_2 + a_3 + a_4 + a_5) = 1 - 1 + 2 - 2 + 503$$

ie. $a_1 + a_2 + a_3 + a_4 + a_5 = 17$.

19. Answer: 54

Multiplying the first three equations by x, y and z respectively, we have

$$xyz = 6\lambda x^2 = 6\lambda y^2 = 6\lambda z^2$$

Since $\lambda \neq 0$ and $x^2+y^2+z^2 = 1$, we deduce that $x^2 = y^2 = z^2 = \frac{1}{3}$, so $x = y = z = \frac{1}{\sqrt{3}}$ and $\lambda = \frac{xyz}{6x^2} = \frac{x^3}{2x^2} = \frac{1}{6\sqrt{3}}$.

Hence

$$(xyz\lambda)^{-1} = (\sqrt{3})^3 6\sqrt{3} = 54$$

20. Answer: 2

Observe that

$$(x+y)(y+z) = xy + xz + y^2 + yz = y(x+y+z) + xz = \frac{1}{xz} + xz \geq 2$$

where the equality holds if and only if $xz = 1$. Let $x = z = 1$ and $y = \sqrt{2} - 1$, then we have the minimum value 2 for $(x+y)(y+z)$.

21. Answer: 2028

Let L_1, L_2, L_3 represent the three lines $y = 4x + 1, y = x + 2$ and $y = -2x + 4$ respectively.

Observe that L_1 and L_2 intersects at $(\frac{1}{3}, \frac{7}{3})$, L_1 and L_3 intersects at $(\frac{1}{2}, 3)$, and L_2 and L_3 intersects at $(\frac{2}{3}, \frac{8}{3})$. Thus

$$f(x) = \begin{cases} 4x + 1, & x < \frac{1}{3}; \\ \frac{7}{3}, & x = \frac{1}{3}; \\ x + 2, & \frac{1}{3} < x < \frac{2}{3}; \\ \frac{8}{3}, & x = \frac{2}{3}; \\ -2x + 4, & x > \frac{2}{3}. \end{cases}$$

Thus the maximum value of $f(x)$ is $\frac{8}{3}$ and the maximum value of $6f(x) + 2012$ is 2028.

22. Answer: 665

Since B cannot be empty, the number of elements in B is between 1 to 6. After picking B with k elements, there are $2^k - 1$ possible subsets of B which qualifies for A, as A and B must be distinct. Thus the total number of possibilities is

$$\begin{aligned} \sum_{k=1}^{6} \binom{6}{k} (2^k - 1) &= \sum_{k=0}^{6} \binom{6}{k} (2^k - 1) \\ &= \sum_{k=0}^{6} \binom{6}{k} 2^k - \sum_{k=0}^{6} \binom{6}{k} \\ &= (2 + 1)^6 - (1 + 1)^6 \\ &= 3^6 - 2^6 \\ &= 665. \end{aligned}$$

23. Answer: 45

The equation can be rewritten as $\sqrt{(\sqrt{x-1} - 2)^2} + \sqrt{(\sqrt{x-1} - 3)^2} = 1$.

If $\sqrt{x-1} \geq 3$, it reduces to $\sqrt{x-1} - 2 + \sqrt{x-1} - 3 = 1$ i.e. $\sqrt{x-1} = 3$ giving $x = 10$.

If $\sqrt{x-1} \leq 2$, it reduces to $2 - \sqrt{x-1} + 3 - \sqrt{x-1} = 1$ i.e. $\sqrt{x-1} = 2$ giving $x = 5$.

If $2 < \sqrt{x-1} < 3$, i.e. $5 < x < 10$, it reduces to $\sqrt{x-1} - 2 + 3 - \sqrt{x-1} = 1$ which is true for all values of x between 5 and 10.

Hence the sum of all integral solutions is $5 + 6 + 7 + 8 + 9 + 10 = 45$.

24. Answer: 4

$$S = \left| \sqrt{(x+2)^2 + (0-1)^2} - \sqrt{(x+1)^2 + (0-2)^2} \right|.$$

Let $P = (x, 0)$, $A = (-2, 1)$ and $B = (-1, 2)$, then S represents the difference between the lengths PA and PB. S is maximum when the points P, A and B are collinear and that occurs when $P = (-3, 0)$. So

$$S = \left|\sqrt{(-1)^2 + (1)^2} - \sqrt{(-2)^2 + (0-2)^2}\right| = \left|\sqrt{2} - 2\sqrt{2}\right|.$$

Thus the maximum value of $S^4 = 4$.

25. Answer: 384

Partition S into three subsets according to their residues modulo 3: $S_0 = \ 3, 6, \ldots, 18\ $, $S_1 = \ 1, 4, \ldots, 19\ $ and $S_2 = \ 2, 5, \ldots, 20\ $. In order for the sum of three integers to be divisible by 3, either all three must belong to exactly one S_i or all three must belong to different S_i.

Hence total number of such choices is $\binom{6}{3} + 2\binom{7}{3} + 6 \times 7 \times 7 = 384$.

26. Answer: 10

Since $BD \parallel FG$ and FG is tangent to the circle at C, we have

$$\angle BCF = \angle CBE = \angle DCG = \angle BDC = \angle BAC.$$

Furthermore

$$\angle BEC = \angle BAC + \angle ABE = \angle CBE + \angle ABE = \angle ABC = \angle ACB.$$

We can then conclude that $BE = BC = DC = 4$. Also, $\triangle ABE$ is similar to $\triangle DCE$. If we let $AE = x$,

$$\frac{DE}{DC} = \frac{AE}{AB} \implies DE = \frac{2}{3}x.$$

By the Intersecting Chord Theorem, $AE \cdot EC = BE \cdot ED$, i.e. $x(6 - x) = 4(\frac{2}{3}x)$, which gives $x = \frac{10}{3}$, so $3AE = 3x = 10$.

27. Answer: 3432

We use $a_1, a_2, \cdots, a_7$ and $b_1, b_2, \cdots, b_7$ to denote the players of Team A and Team B, respectively. A possible gaming outcome can be represented by a linear sequence of the above 14 terms. For instance, we may have $a_1a_2b_1b_2a_3b_3b_4b_5a_4b_6b_7a_5a_6a_7$ which indicates player 1 followed by player 2 from Team A were eliminated first, and the third player eliminated was player 1 from team B. However Team A emerged the final winner as all seven players of Team B gets eliminated with a_5, a_6 and a_7 remaining uneliminated. (Note a_6 and a_7 never actually played.) Thus, a gaming outcome can be formed by choosing 7 out of 14 possible positions for Team A, with the remaining filled by Team B players. Therefore, the total number of gaming outcomes is given by $\binom{14}{7} = \frac{14!}{(7!)^2} = 3432$.

28. Answer: 1

$$\frac{8\sqrt{2}}{5} = \mathbf{m}+\mathbf{n} = \sqrt{4+2\sqrt{2}\cos\theta - 2\sqrt{2}\sin\theta}$$
$$= 2\sqrt{1+\cos\left(\theta+\frac{\pi}{4}\right)}$$
$$= 2\sqrt{2}\left|\cos\left(\frac{\theta}{2}+\frac{\pi}{8}\right)\right|$$

Since $\pi < \theta < 2\pi$, we have $\frac{5}{8}\pi < \frac{\theta}{2}+\frac{\pi}{8} < \frac{9}{8}\pi$. Thus, $\cos\left(\frac{\theta}{2}+\frac{\pi}{8}\right) < 0$. This implies that $\cos\left(\frac{\theta}{2}+\frac{\pi}{8}\right) = -\frac{4}{5}$ and hence

$$5\cos\left(\frac{\theta}{2}+\frac{\pi}{8}\right)+5 = 1$$

29. Answer: 8

Using the AM-GM inequality, we have

$$f(x\ \ y\ \ z) = \sqrt{2x+13}+\sqrt[3]{3y+5}+\sqrt[4]{8z+12}$$
$$= \sqrt{\frac{2x+13}{4}}\cdot\sqrt{4}+\sqrt[3]{\frac{3y+5}{4}}\cdot\sqrt[3]{2}\cdot\sqrt[3]{2}+\sqrt[4]{\frac{8z+12}{8}}\cdot\sqrt[4]{2}\cdot\sqrt[4]{2}\cdot\sqrt[4]{2}$$
$$\leq \frac{\frac{2x+13}{4}+4}{2}+\frac{\frac{3y+5}{4}+2+2}{3}+\frac{\frac{8z+12}{8}+2+2+2}{4}$$
$$= \frac{1}{4}(x+y+z)+\frac{29}{4}$$
$$= 8$$

The equality is achieved at $x = \frac{3}{2}$, $y = 1$ and $z = \frac{1}{2}$.

30. Answer: 40460

Let $n = 34$ and $Q(x) = (x+1)P(x) - x$

Then $Q(x)$ is a polynomial of degree $n+1$ and $Q(k) = 0$ for all $k = 0\ \ 1\ \ 2\ \ \cdots\ \ n$. Thus there is a constant C such that

$$Q(x) = (x+1)P(x) - x = Cx(x-1)(x-2)\cdots(x-n)$$

Letting $x = -1$ gives

$$1 = C(-1)(-2)\cdots(-n-1) = C(-1)^{n+1}(n+1)!$$

Thus $C = (-1)^{n+1}\ \ (n+1)!$ and

$$P(x) = \frac{1}{x+1}(x+Q(x)) = \frac{1}{x+1}\left(x+\frac{(-1)^{n+1}x(x-1)(x-2)\cdots(x-n)}{(n+1)!}\right)$$

So

$$P(n+1) = \frac{1}{n+2}(n+1+\frac{(-1)^{n+1}(n+1)!}{(n+1)!}) = \frac{1}{n+2}(n+1+(-1)^{n+1}) = \frac{n}{n+2}$$

since $n = 34$ is even. Hence

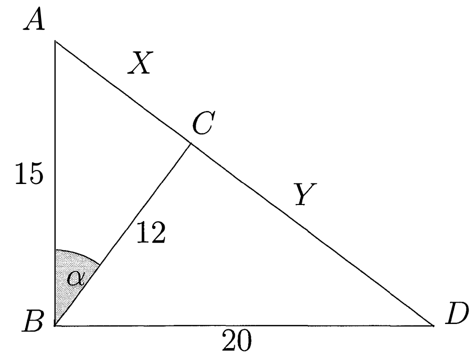
$$42840 \times P(35) = 34 \times 35 \times 36 \times \frac{34}{36} = 40460$$

31. Answer: 30

Let $X = \sqrt{369 - 360\cos\alpha}$ and $Y = \sqrt{544 - 480\sin\alpha}$. Observe that

$$X^2 = 12^2 + 15^2 - 2(12)(15)\cos\alpha$$
$$Y^2 = 12^2 + 20^2 - 2(12)(20)\cos(90° - \alpha)$$

and $15^2 + 20^2 = 25^2 = (X+Y)^2$, so we can construct a right-angled triangle ABD as shown.



In particular $\angle ABC = \alpha$ and $\angle CBD = 90° - \alpha$. We can check that $\triangle ACB$ is in fact similar to $\triangle ABD$. So $\angle ADB = \alpha$ and

$$40\tan\alpha = 40 \times \frac{15}{20} = 30$$

32. Answer: 3

We shall apply the following inequality:

$$4(a^2 + b^2 + c^2 + d^2) \geq (a + b + c + d)^2$$

Since $a + b + c + d = 8 - e$ and $a^2 + b^2 + c^2 + d^2 = 16 - e^2$, we have

$$4(16 - e^2) \geq (8 - e)^2$$

i.e. $e(5e - 16) \leq 0$. Thus $0 \leq e \leq 16\ 5$.

Note that if $a = b = c = d = 6\ 5$, we have $e = 16\ 5$. Hence $\lfloor e \rfloor = 3$.

33. Answer: 105

A three-digit number can be expressed as $100a + 10b + c$, and so we are minimizing

$$F(a\ b\ c) = \frac{100a + 10b + c}{a + b + c}$$

Observe that with distinct digits a b c, $F(a\ b\ c)$ has the minimum value when $a < b < c$. Thus we assume that $0 < a < b < c \leq 9$.

Note that
$$F(a\ b\ c) = \frac{100a + 10b + c}{a + b + c} = 1 + \frac{99a + 9b}{a + b + c}$$
We observe now that $F(a\ b\ c)$ is minimum when $c = 9$.
$$F(a\ b\ 9) = 1 + \frac{99a + 9b}{a + b + 9} = 1 + \frac{9(a + b + 9) + 90a - 81}{a + b + 9} = 10 + \frac{9(10a - 9)}{a + b + 9}$$
Now $F(a\ b\ 9)$ is minimum when $b = 8$.
$$F(a\ 8\ 9) = 10 + \frac{9(10a - 9)}{a + 17} = 10 + \frac{90(a + 17) - 1611}{a + 17} = 100 - \frac{1611}{a + 17}$$
which has the minimum value when $a = 1$, and so $L = F(1\ 8\ 9) = 10\ 5$ and $10L = 105$.

34. Answer: 59

Let $\alpha = 17^{15^{\cdot^{\cdot^{\cdot^{3^1}}}}}$ and $\beta = 15^{13^{\cdot^{\cdot^{\cdot^{3^1}}}}}$. Since $13^{\cdot^{\cdot^{\cdot^{3^1}}}}$ is odd. $\beta \equiv -1 \pmod{16}$. Now let φ be Euler's phi function, $\varphi(100) = 40$ and $\varphi(40) = 16$. By Euler's theorem,
$$\alpha = 17^{\beta} \equiv 17^{-1} \equiv 33 \pmod{40}$$
where the last congruence can be calculated by the extended Euclidean algorithm. Thus by repeated squaring, we have
$$19^{\alpha} \equiv 19^{33} \equiv 59 \pmod{100}$$

35. Answer: 5

Note that $(n + \frac{1}{2})^2 = n^2 + n + \frac{1}{4}$, so $f(n^2 + n) = n$ but $f(n^2 + n + 1) = n + 1$. So each of the sequences $(n - f(n))_{n=1}^{\infty} = (0\ 1\ 1\ 2\quad)$ and $(n + f(n))_{n=1}^{\infty} = (2\ 3\ 5\ 6\quad)$ increases by 1 for every increment of n by 1, except when $n = m^2 + m$. If $n = m^2 + m$, we have $n - f(n) = m^2$ and $(n + 1 - f(n + 1)) = m^2$, so the former sequence has every perfect square repeated once. On the other hand, if $n = m^2 + m$, we have $n + f(n) = m^2 + 2m$ but $(n + 1 + f(n + 1)) = m^2 + 2m + 2$, so the latter sequence omits every perfect square. Thus
$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\left(\frac{3}{2}\right)^{f(n)} + \left(\frac{3}{2}\right)^{-f(n)}}{\left(\frac{3}{2}\right)^{n}} &= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-f(n)} + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n+f(n)} \\ &= \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n} + \sum_{m=1}^{\infty} \left(\frac{2}{3}\right)^{m^2} + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n} - \sum_{m=1}^{\infty} \left(\frac{2}{3}\right)^{m^2} \\ &= 5\end{aligned}$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

(Senior Section, Round 2)

Saturday, 23 June 2012 **0900-1300**

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1. A circle ω through the incentre I of a triangle ABC and tangent to AB at A, intersects the segment BC at D and the extension of BC at E. Prove that the line IC intersects ω at a point M such that $MD = ME$.

2. Determine all positive integers n such that n equals the square of the sum of the digits of n.

3. If 46 squares are colored red in a 9×9 board, show that there is a 2×2 block on the board in which at least 3 of the squares are colored red.

4. Let $a_1 \ a_2 \qquad a_n \ a_{n+1}$ be a finite sequence of real numbers satisfying

$$\begin{aligned} & a_0 = a_{n+1} = 0 \\ \text{and} \quad & a_{k-1} - 2a_k + a_{k+1} \le 1 \quad \text{for } k = 1 \ 2 \qquad n \end{aligned}$$

Prove that for $k = 0 \ 1 \qquad n+1$,

$$a_k \le \frac{k(n+1-k)}{2}$$

5. Prove that for any real numbers $a \ b \ c \ d \geq 0$ with $a + b = c + d = 2$,

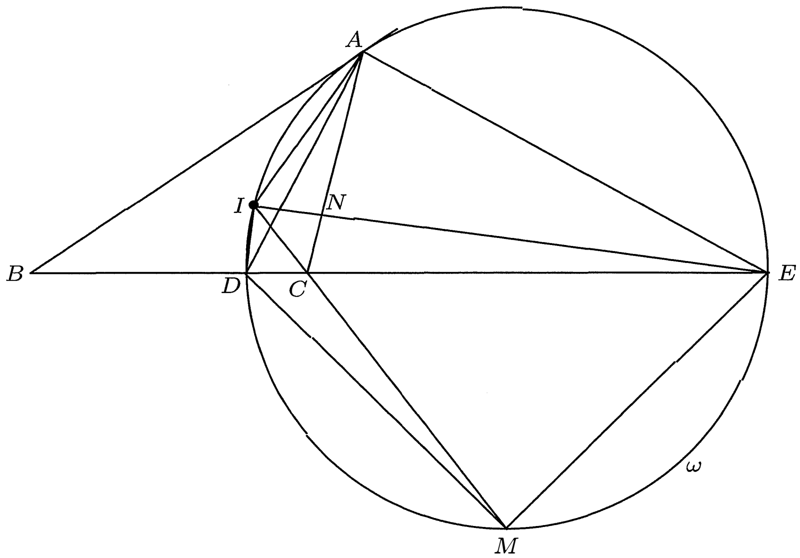
$$(a^2 + c^2)(a^2 + d^2)(b^2 + c^2)(b^2 + d^2) \le 25$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

(Senior Section, Round 2 solutions)

1. Join AD, ID, IA and AE. Let IE intersect AC at N. We have $\angle IAN = \angle IAB = \angle IEA$ so that the triangles NIA and AIE are similar. Thus $\angle ANI = \angle EAI = \angle IDB$. Also $\angle DCI = \angle NCI$. Therefore, the triangles DCI and NCI are congruent. Hence $\angle DIC = \angle NIC$ implying $MD = ME$.



2. Let $s(n)$ denote the sum of all the digits of n. Suppose n is a positive integer such that $s(n)^2 = n$. Let $s(n) = k$ so that $n = k^2$. Then $s(k^2) = s(n) = k$. Let $10^{r-1} \leq k < 10^r$, where r is a positive integer. That is k has exactly r digits. From $10^{r-1} \leq k$, we have $r \leq \log k + 1$. From $k < 10^r$, we have $k^2 < 10^{2r}$ so that k^2 has at most $2r$ digits. Therefore, $s(k^2) \leq 9 \times 2r = 18r \leq 18 \log k + 18$ which is less than k if $k \geq 50$. Thus the equation $s(k^2) = k$ has no solution in k if $k \geq 50$.

Let $k < 50$ and $s(k^2) = k$. Taking mod 9, we get $k^2 \equiv k \pmod 9$. Thus $k \equiv 0, 1 \pmod 9$. That is $k = 1, 9, 10, 18, 19, 27, 28, 36, 37, 45, 46$. Only when $k = 1$ and $k = 9$, we have $s(k^2) = k$. The corresponding solutions for n are $n = 1$ or 81.

3. Suppose that at most 2 squares are colored red in any 2×2 square. Then in any 9×2 block, there are at most 10 red squares. Moreover, if there are 10 red squares, then there must be 5 in each row.

Now let the number of red squares in row i of the 9×9 board be r_i. Then $r_i + r_{i+1} \leq 10$, $1 \leq i \leq 8$. Suppose that some $r_i \leq 5$ with i odd. Then

$$(r_1 + r_2) + \cdots + (r_{i-2} + r_{i-1}) + r_i + \cdots + (r_8 + r_9) \leq 4 \times 10 + 5 = 45.$$

On the other hand, suppose that $r_1, r_3, r_5, r_7, r_9 \geq 6$. Then the sum of any 2 consecutive r_i's is ≤ 9. So

$$(r_1 + r_2) + \cdots + (r_7 + r_8) + r_9 \leq 4 \times 9 + 9 = 45.$$

4. Let $b_k = \frac{k(n+1-k)}{2}$. Then $b_0 = b_{n+1} = 0$ and $b_{k-1} - 2b_k + b_{k+1} = -1$ for $k = 1, 2 \ldots, n$. Suppose there exists an index i such that $a_i > b_i$, then the sequence $a_0 - b_0, \ldots, a_{n+1} - b_{n+1}$ has a positive term. Let j be the index such that $a_{j-1} - b_{j-1} < a_j - b_j$ and $a_j - b_j$ has the largest value. Then

$$(a_{j-1} - b_{j-1}) + (a_{j+1} - b_{j+1}) < 2(a_j - b_j).$$

Using

$$a_{k-1} - 2a_k + a_{k+1} \geq -1 \quad \text{and} \quad b_{k-1} - 2b_k + b_{k+1} = -1 \quad \text{for all } k$$

we obtain

$$(a_{j-1} - b_{j-1}) + (a_{j+1} - b_{j+1}) \geq 2(a_j - b_j)$$

a contradiction. Thus $a_k \leq b_k$ for all k. Similarly, we can show that $a_k \geq -b_k$ for all k and therefore $a_k \leq b_k$ as required.

5. First note that $(ac + bd)(ad + bc) \geq (ab - cd)^2$. To see this, we may assume $a \geq c \geq d \geq b$ since $a + b = c + d$. Then $cd - ab \geq 0$. Also we have the two obvious inequalities $ac + bd \geq cd - ab$ and $ad + bc \geq cd - ab$. Multiplying them together we get $(ac + bd)(ad + bc) \geq (ab - cd)^2$. Next

$$\begin{aligned}
&(a^2 + c^2)(a^2 + d^2)(b^2 + c^2)(b^2 + d^2) \\
&\quad = \left((ac + bd)(ad + bc) - (ab - cd)^2\right)^2 + (ab - cd)^2 \\
&\qquad + \left((a + b)^2(c + d)^2 - 1\right)(ab - cd)^2 \\
&\quad = \left((ac + bd)(ad + bc) - (ab - cd)^2\right)^2 + 16(ab - cd)^2 \\
&\quad \leq \left(\frac{(ac + bd + ad + bc)^2}{4} - (ab - cd)^2\right)^2 + 16(ab - cd)^2 \quad \text{by AM-GM} \\
&\quad = \left(\frac{(a + b)^2(c + d)^2}{4} - (ab - cd)^2\right)^2 + 16(ab - cd)^2 \\
&\quad = \left(4 - (ab - cd)^2\right)^2 + 16(ab - cd)^2.
\end{aligned}$$

This final expression is an increasing function of $(ab - cd)^2$. The largest value of $(ab - cd)^2$ is 1 when $(a, b, c, d) = (1, 1, 0, 2), (1, 1, 2, 0), (0, 2, 1, 1), (2, 0, 1, 1)$. Consequently, $\left(4 - (ab - cd)^2\right)^2 + 16(ab - cd)^2 \leq 25$, proving the inequality.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2012
(Open Section, First Round)

Wednesday, 30 May 2012 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Throughout this paper, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$ (This notation is used in Questions 2, 10, 16, 17, 18 and 22).

1. The sum of the squares of 50 consecutive odd integers is 300850. Find the largest odd integer whose square is the last term of this sum.

2. Find the value of $\sum_{k=3}^{1000} \lfloor \log_2 k \rfloor$.

3. Given that $\mathrm{f}(x)$ is a polynomial of degree 2012, and that $\mathrm{f}(k) = \frac{2}{k}$ for $k = 1, 2, 3, \cdots, 2013$, find the value of $2014 \times \mathrm{f}(2014)$.

4. Find the total number of sets of positive integers (x, y, z), where x, y and z are positive integers, with $x < y < z$ such that
$$x + y + z = 203.$$

5. There are a few integers n such that $n^2 + n + 1$ divides $n^{2013} + 61$. Find the sum of the squares of these integers.

6. It is given that the sequence $(a_n)_{n=1}^{\infty}$, with $a_1 = a_2 = 2$, is given by the recurrence relation
$$\frac{2a_{n-1}a_n}{a_{n-1}a_{n+1} - a_n^2} = n^3 - n$$
for all $n = 2, 3, 4, \cdots$. Find the integer that is closest to the value of $\sum_{k=2}^{2011} \frac{a_{k+1}}{a_k}$.

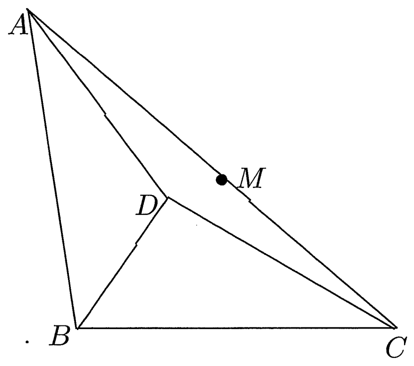
7. Determine the largest even positive integer which cannot be expressed as the sum of two composite odd positive integers.

8. The lengths of the sides of a triangle are successive terms of a geometric progression. Let A and C be the smallest and largest interior angles of the triangle respectively. If the shortest side has length 16 cm and
$$\frac{\sin A - 2\sin B + 3\sin C}{\sin C - 2\sin B + 3\sin A} = \frac{19}{9},$$
find the perimeter of the triangle in centimetres.

9. Find the least positive integral value of n for which the equation
$$x_1^3 + x_2^3 + \cdots + x_n^3 = 2002^{2002}$$
has integer solutions $(x_1, x_2, x_3, \cdots, x_n)$.

10. Let α_n be a real root of the cubic equation $nx^3 + 2x - n = 0$, where n is a positive integer. If $\beta_n = \lfloor (n+1)\alpha_n \rfloor$ for $n = 2\ 3\ 4\ \cdots$, find the value of $\dfrac{1}{1006}\displaystyle\sum_{k=2}^{2013} \beta_k$.

11. In the diagram below, the point D lies inside the triangle ABC such that $\angle BAD = \angle BCD$ and $\angle BDC = 90^{\circ}$. Given that $AB = 5$ and $BC = 6$, and the point M is the midpoint of AC, find the value of $8 \times DM^2$.



12. Suppose the real numbers x and y satisfy the equations

$$x^3 - 3x^2 + 5x = 1 \quad \text{and} \quad y^3 - 3y^2 + 5y = 5$$

Find $x + y$.

13. The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32. Determine the value of k.

14. Determine the smallest integer n with $n \geq 2$ such that

$$\sqrt{\frac{(n+1)(2n+1)}{6}}$$

is an integer.

15. Given that f is a real-valued function on the set of all real numbers such that for any real numbers a and b,

$$\mathrm{f}(a\mathrm{f}(b)) = ab$$

Find the value of $\mathrm{f}(2011)$.

16. The solutions to the equation $x^3 - 4\lfloor x \rfloor = 5$, where x is a real number, are denoted by $x_1\ x_2\ x_3\ \cdots x_k$ for some positive integer k. Find $\displaystyle\sum_{i=1}^{k} x_i^3$.

17. Determine the maximum integer solution of the equation

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \cdots + \left\lfloor \frac{x}{10!} \right\rfloor = 1001$$

18. Let A, B, C be the three angles of a triangle. Let L be the maximum value of

$$\sin 3A + \sin 3B + \sin 3C\ .$$

Determine $\lfloor 10L \rfloor$.

19. Determine the number of sets of solutions (x, y, z), where x, y and z are integers, of the the equation $x^2 + y^2 + z^2 = x^2y^2$.

20. We can find sets of 13 distinct positive integers that add up to 2142. Find the largest possible greatest common divisor of these 13 distinct positive integers.

21. Determine the maximum number of different sets consisting of three terms which form an arithmetic progressions that can be chosen from a sequence of real numbers $a_1, a_2, \cdots, a_{101}$, where

$$a_1 < a_2 < a_3 < \cdots < a_{101}.$$

22. Find the value of the series

$$\sum_{k=0}^{\infty} \left\lfloor \frac{20121 + 2^k}{2^{k+1}} \right\rfloor .$$

23. The sequence $(x_n)_{n=1}^{\infty}$ is defined recursively by

$$x_{n+1} = \frac{x_n + (2 - \sqrt{3})}{1 - x_n(2 - \sqrt{3})},$$

with $x_1 = 1$. Determine the value of $x_{1001} - x_{401}$.

24. Determine the maximum value of the following expression

$$- \cdots - x_1 - x_2 - x_3 - x_4 - \cdots - x_{2014} -$$

where $x_1, x_2, \cdots, x_{2014}$ are distinct numbers in the set $-1, 2, 3, 4, \cdots, 2014-$.

25. Evaluate $\dfrac{-1}{2^{2011}} \displaystyle\sum_{k=0}^{1006} (-1)^k 3^k \binom{2012}{2k}$.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2012
(Open Section, First Round Solutions)

1. Answer: 121

   Solution. Let the integers be $X+2\ X+4\ \cdots\ X+100$. Then

$$(X+2)^2+(X+4)^2+\cdots+(X+100)^2=300850$$

   Let $y=X+51$ and regrouping the terms, we obtain

$$\left[(y-49)^2+(y+49)^2\right]+\left[(y-47)^2+(y+47)^2\right]+\cdots+\left[(y-1)^2+(y+1)^2\right]=300850$$

   which simplifies to

$$50y^2+2(1^2+3^2+5^2+7^2+\cdots+49^2)=300850$$

   Using the fact that $1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{4}{3}n^3-\frac{1}{3}n$, we obtain $y=72$. Hence $X=21$, so that the required number is 121.

   □

2. Answer: 7986

   Solution. Note that $2^{k+1}-2^k=2^k$, and that $2^k\le t<2^{k+1}$ if and only if $\lfloor\log_2 t\rfloor=k$. Then the requires sum (denoted by S) can be obtained by

$$\begin{aligned} S &= \sum_{k=2}^{9}\sum_{2^k\le t<2^{k+1}}\lfloor\log_2 t\rfloor+\lfloor\log_2 3\rfloor-\sum_{t=1001}^{1023}\lfloor\log_2 t\rfloor \\ &= \left(\sum_{k=2}^{9}k2^k\right)+1-\sum_{t=1001}^{1023}9 \\ &= 8192+1-23(9)=7986 \end{aligned}$$

   □

3. Answer: 4

   Solution. Let $\mathrm{g}(x)=x\mathrm{f}(x)-2$, hence $\mathrm{g}(x)$ is a polynomial of degree 2013. Since $\mathrm{g}(1)=\mathrm{g}(2)=\mathrm{g}(3)=\cdots=\mathrm{g}(2013)=0$, we must have

$$\mathrm{g}(x)=\lambda(x-1)(x-2)(x-3)\cdots(x-2012)(x-2013)$$

   for some λ. Also, $\mathrm{g}(0)=-2=-\lambda\cdot 2013!$, we thus have $\lambda=\frac{2}{2013!}$ Hence,

$$\mathrm{g}(2014)=\frac{2}{2013!}(2013!)=2014\cdot\mathrm{f}(2014)-2$$

   concluding that $2014\cdot\mathrm{f}(2014)=4$ □

4. Answer: 3333

Solution. First note that there are $\binom{202}{2} = \dfrac{202(201)}{2} = 20301$ positive integer sets (x, y, z) which satisfy the given equation. These solution sets include those where two of the three values are equal. If $x = y$, then $2x+z = 203$. By enumerating, $z = 1, 3, 5, \cdots, 201$. There are thus 101 solutions of the form (x, x, z). Similarly, there are 101 solutions of the form (x, y, x) and (x, y, y). Since $x < y < z$, the required answer is

$$\frac{1}{3!}\left(\binom{202}{2} - 3(101)\right) = \frac{20301 - 303}{6} = 3333.$$

□

5. Answer: 62

Solution. Since $n^3 - 1 = (n-1)(n^2+n+1)$, we know that n^2+n+1 divides n^3-1. Also, since $n^{2013} - 1 = (n^3)^{671} - 1$, we also know that n^2+n+1 divides $n^{2013}-1$. As

$$n^{2013} + 61 = n^{2013} - 1 + 62,$$

we must have that n^2+n+1 divides $n^{2013}+61$ if and only if n^2+n+1 divides 62.

Case (i): If $n^2+n+1 = 1$, then $n = 0, -1$.

Case (ii): If $n^2+n+1 = 2$, there is no integer solution for n.

Case (iii): If $n^2+n+1 = 31$, then $n = 6, -5$.

Case (iv): If $n^2+n+1 = 62$, there is no integer solution for n.

Thus, all the integer values of n are $0, -1, 6, -5$. Hence the sum of squares is $1+36+25 = 62$.

□

6. Answer: 3015

Solution. The recurrence relation can be written as

$$\frac{a_{n+1}}{a_n} - \frac{a_n}{a_{n-1}} = \left(\frac{1}{n-1} - \frac{1}{n}\right) - \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Summing for $n = 2$ to N, we obtain

$$\frac{a_{N+1}}{a_N} - \frac{a_2}{a_1} = \left(1 - \frac{1}{N}\right) - \left(\frac{1}{2} - \frac{1}{N+1}\right),$$

showing that

$$\frac{a_{N+1}}{a_N} = \frac{3}{2} - \left(\frac{1}{N} - \frac{1}{N+1}\right).$$

Summing this up for $N = 2$ to $N = 2011$, we obtain

$$S = \sum_{k=2}^{2011} \frac{a_{k+1}}{a_k} = \frac{3}{2}(2010) - \left(\frac{1}{2} - \frac{1}{2012}\right) = 3014.5 + \frac{1}{2012}$$

showing that the integer closest to S is 3015. □

7. Answer: 38

Solution. Let n be an even positive integer. Then each of the following expresses n as the sum of two odd integers: $n = (n-15)+15$ $(n-25)+25$ or $(n-35)+35$. Note that at least one of $n-15$ $n-25$ $n-35$ is divisible by 3 so that it is composite, and hence n can be expressed as the sum of two composite odd numbers if $n > 38$. Indeed, it can be verified that 38 cannot be expressed as the sum of two composite odd positive integers. □

8. Answer: 76

Solution. Let the lengths of the sides of the triangle in centimetres be 16, $16r$ and $16r^2$ (where $r > 1$). Then $\dfrac{1-2r+3r^2}{r^2-2r+3} = \dfrac{19}{9}$ so that $r = \dfrac{3}{2}$. Hence, the perimeter of the triangle $= 16\left(1+\dfrac{3}{2}+\dfrac{9}{4}\right) = 76\text{cm}$ □

9. Answer: 4

Solution. Since $2002 \equiv 4(\text{mod}9)$ $4^3 \equiv 1(\text{mod}9)$ and $2002 = 667\times 3+1$, it follows that $2002^{2002} \equiv 4^{667\times 3+1} \equiv 4(\text{mod}9)$. Observe that for positive integers x, the possible residues modulo 9 for x^3 are 0 ± 1 . Therefore, none of the following

$$x_1^3 \quad x_1^3+x_2^3 \quad x_1^3+x_2^3+x_3^3$$

can have a residue of 4 modulo 9. However, since $2002 = 10^3+10^3+1^3+1^3$, it follows that

$$\begin{aligned} 2002^{2002} &= 2002\cdot(2002^{667})^3 \\ &= (10^3+10^3+1^3+1^3)(2002^{667})^3 \\ &= (10\cdot 2002^{667})^3+(10\cdot 2002^{667})^3+(2002^{667})^3+(2002^{667})^3 \end{aligned}$$

This shows that $x_1^3+x_2^3+x_3^3+x_4^4 = 2002^{2002}$ is indeed solvable. Hence the least integral value of n is 4. □

10. Answer: 2015

Let $\text{f}(x) = nx^3+2x-n$. It is easy to see that f is a strictly increasing function for $n = 2\ 3\ 4\ \cdots$. Further,

$$f\left(\frac{n}{n+1}\right) = n\left(\frac{n}{n+1}\right)^3+2\left(\frac{n}{n+1}\right)-n = \frac{n}{(n+1)^3}\left(-n^2+n+1\right) < 0$$

for all $n \geq 2$. Also, $\text{f}(1) = 2 > 0$. Thus, the only real root of the equation $nx^3+2x-n = 0$ for $n \geq 2$ is located in the interval $\left(\frac{n}{n+1}\ 1\right)$. We must have

$$\frac{n}{n+1} < \alpha_n < 1 \Longrightarrow n < (n+1)\alpha_n < n+1$$

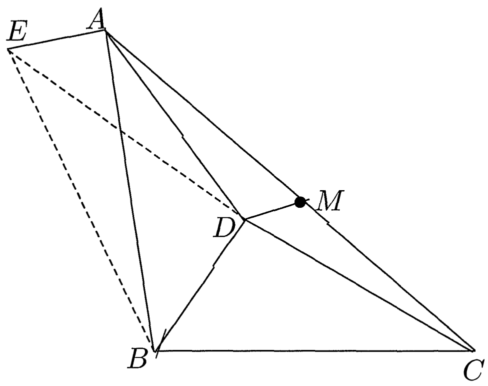
so that $\beta_n = \lfloor (n+1)\alpha_n \rfloor = n$ for all $n \geq 2$. Consequently,

$$\frac{1}{1006}\sum_{k=2}^{2013}\beta_k = \frac{1}{1006}\sum_{k=2}^{2013}k = \frac{1}{1006}\cdot\frac{2012}{2}(2+2013) = 2015$$

□

11. Answer: 22

Solution. Extend CD to E such that $CD = DE$.



It is clear that ΔCDB and ΔEDB are congruent. Hence $EB = CB = 6$ and $\angle BED = \angle BCD$. Thus, $\angle BED = \angle BCD = \angle BAD$ implies that the points B, D, A are E are concyclic. Given that $\angle BDC = 90^\circ$, hence $\angle EDB = 90^\circ$. $BDAE$ is a cyclic quadrilateral with EB as a diameter. Thus, $\angle EAB = 90^\circ$. In the right-angled triangle EAB, we have

$$AE = \sqrt{EB^2 - AB^2} = \sqrt{6^2 - 5^2} = \sqrt{11}.$$

Since D and M are the midpoints of EC and AC respectively, $DM = \frac{1}{2}AE = \frac{\sqrt{11}}{2}$. Thus, $8 \times DM^2 = 22$. □

12. Answer: 2

Solution. From $x^3 - 3x^2 + 5x = 1$, we have

$$(x-1)^3 + 2(x-1) = -2,$$

and from $y^3 - 3y^2 + 5y = 5$, we have

$$(y-1)^3 + 2(y-1) = 2.$$

Thus

$$\begin{aligned} 0 &= (x-1)^3 + 2(x-1) + (y-1)^3 + 2(y-1) \\ &= (x+y-2)((x-1)^2 - (x-1)(y-1) + (y-1)^2) + 2(x+y-2) \\ &= (x+y-2)(2 + (x-1)^2 - (x-1)(y-1) + (y-1)^2). \end{aligned}$$

For any real numbers x and y, we always have

$$(x-1)^2 - (x-1)(y-1) + (y-1)^2 \geq 0$$

and thus $x + y - 2 = 0$, implying that $x + y = 2$. □

13. Answer: 86

Solution. Let a, b, c, d be the four roots of $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ such that $ab = -32$. Then

$$\begin{cases} a+b+c+d = 18, \\ ab+ac+ad+bc+bd+cd = k, \\ abc+abd+acd+bcd = -200, \\ abcd = -1984. \end{cases}$$

Since $ab = -32$ and $abcd = -1984$, we have $cd = 62$. Then, from $abc+abd+acd+bcd = -200$ we have

$$-200 = -32c - 32d + 62a + 62b = -32(c+d) + 62(a+b).$$

Solving this equation together with the equation $a+b+c+d = 18$ gives that

$$a+b = 4, \qquad c+d = 14.$$

From $ab+ac+ad+bc+bd+cd = k$, we have

$$k = ab+ac+ad+bc+bd+cd = -32+ac+ad+bc+bd+62$$
$$= 30 + (a+b)(c+d) = 86.$$

□

14. Answer: 337

Solution. Assume that

$$\sqrt{\frac{(n+1)(2n+1)}{6}} = m$$

and so

$$(n+1)(2n+1) = 6m^2.$$

Thus 6 $(n+1)(2n+1)$, implying that $n \equiv 1$ or $5 (\text{mod } 6)$.

Case 1: $n = 6k+5$.

Then $m^2 = (k+1)(12k+11)$. Since $(k+1)$ and $(12k+11)$ are relatively prime, both must be squares. So there are positive integers a and b such that $k+1 = a^2$ and $12k+11 = b^2$.

Thus $12a^2 = b^2 + 1$. But, as $12a^2 \equiv 0 (\text{mod } 4)$ and $b^2 + 1 \equiv 1$ or $2 (\text{mod } 4)$, there are no integers a and b such that $12a^2 = b^2 + 1$. Hence Case 1 cannot happen.

Case 2: $n = 6k+1$.

Then $m^2 = (3k+1)(4k+1)$. Since $3k+1$ and $4k+1$ are relatively prime, both must be squares. So there are positive integers a and b such that $3k+1 = a^2$ and $4k+1 = b^2$. Then $3b^2 = (2a-1)(2a+1)$. Observe that in the left-hand side, every prime factor except 3 has an even power. So neither $2a-1$ nor $2a+1$ can be a prime other than 3.

Now we consider positive integers a such that neither $2a-1$ nor $2a+1$ can be a prime other than 3. If $a = 1$, then $b = 1$ and $n = 1$. So we consider $a \geq 2$. The next smallest suitable value for a is 13. When $a = 13$, we have

$$3b^2 = 25 \times 27$$

and so $b = 15$, implying that $k = 56$ and so $n = 6k+1 = 337$. □

15. Answer: 2011

From the recurrence relation, $f(f(1)f(b)) = f(1)b$ and $f(f(b)f(1)) = f(b) \cdot 1$. Hence, $f(b) = f(1)b$. By letting $b = f(1)$, we obtain $f(f(1)) = (f(1))^2$. Also, from the given functional equation, we have $f(f(1)) = 1$, hence $(f(1))^2 = 1$, following that $f(1)$ is either 1 or -1. Hence $f(2011) = 2011$. □

16. Answer: 19

Solution. Note that $x - 1 < \lfloor x \rfloor \le x$. Note that if $x \ge 3$, there will be no solution as

$$x^3 - 4\lfloor x \rfloor \ge x^3 - 4x = x(x^2 - 4) \ge 3(5) = 15.$$

Also, if $x \le -2$, there will be no solution as $x^3 - 4\lfloor x \rfloor < x^3 - 4(x-1) = x(x^2-4)+4 \le 4$. Hence the solution must be in the interval $(-2, 3)$.

If $\lfloor x \rfloor = -2$, then $x^3 = -3$, giving $x = \sqrt[3]{-3}$, which is a solution.

If $\lfloor x \rfloor = -1$, then $x^3 = 1$, giving $x = 1$ which contradicts with $\lfloor x \rfloor = -1$.

If $\lfloor x \rfloor = 0$, then $x^3 = 5$, hence there is no solution.

If $\lfloor x \rfloor = 1$, then $x^3 = 9$. Since $2 < \sqrt[3]{9} < 3$, there is no solution.

If $\lfloor x \rfloor = 2$, then $x^3 = 13$. Since $2 < \sqrt[3]{13} < 3$, $x = \sqrt[3]{13}$ is a solution.

Thus, the required answer is $-3 + 13 = 10$. □

17. Answer: 584

Solution. It is clear that

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \cdots + \left\lfloor \frac{x}{10!} \right\rfloor$$

is a monotone increasing function of x, and when $x = 6!$, the above expression has a value larger than 1001. Thus each solution of the equation

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \cdots + \left\lfloor \frac{x}{10!} \right\rfloor = 1001$$

is less than 6! and so if x is a solution, then

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \cdots + \left\lfloor \frac{x}{10!} \right\rfloor = \left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \left\lfloor \frac{x}{4!} \right\rfloor + \left\lfloor \frac{x}{5!} \right\rfloor.$$

As $x < 6!$, it has a unique expression of the form

$$x = a \times 5! + b \times 4! + c \times 3! + d \times 2! + e,$$

where a, b, c, d, e are non-negative integer with $a \le 5, b \le 4, c \le 3, d \le 2, e \le 1$. Note that if

$$x = a \times 5! + b \times 4! + c \times 3! + d \times 2! + e,$$

then

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \left\lfloor \frac{x}{4!} \right\rfloor + \left\lfloor \frac{x}{5!} \right\rfloor = 206a + 41b + 10c + 3d + e.$$

Since $41b + 10c + 3d + e \le 201$, we have $800 \le 206a \le 1001$ and so $a = 4$. Thus

$$41b + 10c + 3d + e = 177,$$

which implies that $b = 4$ and so on, giving that $c = d = 1$ and $e = 0$. Thus

$$x = 4 \times 5! + 4 \times 4! + 1 \times 3! + 1 \times 2! + 0 = 584.$$

As 584 is the only integer solution, the answer is 584.

$\square$

18. Answer: 25

Solution. We shall show that $-2 \le \sin 3A + \sin 3B + \sin 3C \le 3\sqrt{3}/2$.

Assume that $A \le B \le C$. Then $A \le 60^o$. Thus $\sin 3A \ge 0$. It is clear that $\sin 3B \ge -1$ and $\sin 3C \ge -1$. Thus $\sin 3A + \sin 3B + \sin 3C \ge -2$. Let $B = C$. Then $B = C = 90^o - A/2$. If A is very small, B and C are close to 90^o, and thus $\sin 3A + \sin 3B + \sin 3C$ is close to -2.

Now we show that $\sin 3A + \sin 3B + \sin 3C \le 3\sqrt{3}/2$. First the upper bound can be reached when $A = B = 20^o$ and $C = 140^o$.

Let $X = 3A$, $Y = 3B$ and $Z = 3(C - 120^o)$. Then $X + Y + Z = 180^o$ and

$$\sin 3A + \sin 3B + \sin 3C = \sin X + \sin Y + \sin Z.$$

Suppose that X, Y, Z satisfy the condition that $X + Y + Z = 180^o$ such that $\sin X + \sin Y + \sin Z$ has the maximum value. We can then show that $X = Y = Z$.

Assume that $X \le Y \le Z$. If $X < Z$, then

$$\sin X + \sin Z = 2 \sin \frac{X+Z}{2} \cos \frac{X-Z}{2} < 2 \sin \frac{X+Z}{2},$$

implying that

$$\sin X + \sin Y + \sin Z < \sin \frac{X+Z}{2} + \sin Y + \sin \frac{X+Z}{2}$$

which contradicts the assumption that $\sin X + \sin Y + \sin Z$ has the maximum value. Hence $X = Y = Z = 60^o$, implying that $A = 20^o$, $B = 20^o$ and $C = 140^o$ and

$$\sin 3A + \sin 3B + \sin 3C = 3\sqrt{3}/2.$$

Since $\sqrt{3} \approx 1.732$, the answer is then obtained. $\square$

19. Answer: 1

Solution. Note that $x = 0, y = 0$ and $z = 0$ is a solution of this equation. We shall show that this is its only integer solution by proving that if x, y, z is a solution of this equation and whenever x, y, z are divisible by 2^k, they are also divisible by 2^{k+1} for any $k \ge 0$.

Let $x = 2^k x'$, $y = 2^k y'$ and $z = 2^k z'$. Then $x^2 + y^2 + z^2 = x^2 y^2$ is changed to

$$x'^2 + y'^2 + z'^2 = 2^k x'^2 y'^2$$

It is easy to verify that only when x', y', z' are all even, $x'^2 + y'^2 + z'^2$ and $2^{2k} x'^2 y'^2$ have the same remainder when divided by 4. Thus x, y, z are divisible by 2^{k+1}. $\square$

20. Answer: 21

Solution. Let d be the greatest common divisor (gcd) of these 13 distinct positive integers. Then these 13 integers can be represented as $da_1, da_2, \cdots, da_{13}$, where $\gcd(a_1, a_2, \cdots, a_{13}) = 1$. Let S denote $a_1 + a_2 + \cdots + a_{13}$. Then $Sd = 2142$. In order for d to be the largest possible, S must be the smallest. Since $S \geq 1+2+3+\cdots+13 = 91$ and that S divides 2142, and that $2142 = 2 \times 3 \times 7 \times 51$, the smallest possible value of S can be $2 \times 51 = 102$, and the largest value of d is thus 21. By choosing $(a_1, a_2, a_3, \cdots, a_{12}, a_{13}) = (1, 2, 3, \cdots, 12, 24)$, we conclude that $d = 21$ is possible. $\square$

21. Answer: 2500

Solution. First, for the following particular sequence, there are really 2500 different three-term arithmetic progressions which can be chosen from this sequence:

$$1, 2, 3, \cdots, 101.$$

They are $s, i, 2i - s$ for all integers s, i with $1 \leq s < i \leq 51$ and $2i - t, i, t$ for all integers i and t with $52 \leq i < t \leq 101$.

Now we show that for any given sequence of real numbers $a_1 < a_2 < \cdots < a_{101}$, there are at most 2500 different three-term arithmetic progressions which can be chosen from this sequence.

Let a_s, a_i, a_t represent a three-term arithmetic progression. It is clear that $2 \leq i \leq 100$. If $2 \leq i \leq 51$, then the first term a_s has at most $i - 1$ choices, as s must be an index in $1, 2, \cdots, i-1$. If $52 \leq i \leq 100$, then the third term a_t has at most $101 - i$ choices, as t must be an index in $i+1, i+2, \cdots, 101$.

So the number of different three-term arithmetic progressions which can be chosen from this sequence is at most

$$\sum_{i=2}^{51}(i-1) + \sum_{i=52}^{100}(101-i) = 1 + 2 + \cdots + 50 + 1 + 2 + \cdots + 49 = 2500.$$

$\square$

22. Answer: 20121

Solution. Write $x := x - \lfloor x \rfloor$. Then

$$\left\lfloor \frac{1}{2} + \left\{\frac{x}{2}\right\} \right\rfloor = \begin{cases} 0, & \text{if } \frac{x}{2} < \frac{1}{2} \\ 1, & \text{otherwise} \end{cases} = \left\lfloor 2\left\{\frac{x}{2}\right\} \right\rfloor.$$

Thus, we have

$$\left\lfloor \frac{1}{2} + \frac{x}{2} \right\rfloor = \left\lfloor 2\left\{\frac{x}{2}\right\} \right\rfloor + \left\lfloor \frac{x}{2} \right\rfloor = \lfloor x \rfloor - \left\lfloor \frac{x}{2} \right\rfloor.$$

Applying the above result for $x = \frac{n}{2^k}$,

$$\begin{aligned} \sum_{k=0}^{\infty} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor &= \sum_{k=0}^{\infty} \left(\left\lfloor \frac{n}{2^k} \right\rfloor - \left\lfloor \frac{n}{2^{k+1}} \right\rfloor \right) \\ &= \left\lfloor \frac{n}{2^0} \right\rfloor \\ &= n. \end{aligned}$$

In particular, when $n = 20121$, the infinite series converges to 20121. $\square$

23. Answer: 0

Solution. Let $x_n = \tan \alpha_n$. Since $2 - \sqrt{3} = \tan\left(\frac{\pi}{12}\right)$, it follows that

$$x_{n+1} = \tan \alpha_{n+1} = \frac{\tan \alpha_n + \tan\left(\frac{\pi}{12}\right)}{1 - \tan \alpha_n \tan\left(\frac{\pi}{12}\right)} = \tan\left(\alpha_n + \frac{\pi}{12}\right)$$

So, $x_{n+12} = \tan(\alpha_n + \pi) = \tan \alpha_n = x_n$, implying that this sequence has a period of 12. Observe that $1001 \equiv 5 \pmod{12}$ and $401 \equiv 5 \pmod{12}$. Consequently,

$$x_{1001} - x_{401} = x_5 - x_5 = 0$$

□

24. Answer: 2013

Solution. First it is clear that the answer is an integer between 0 and 2014. But it cannot be 2014, as $\cdots \quad x_1 - x_2 \ - x_3 \ - x_4 \ \cdots - x_{2014}$ and

$$x_1 + x_2 + \cdots + x_{2014} = 1 + 2 + \cdots + 2014 = 1007 \times 2015$$

have the same parity.

Now we just need to show that 2013 can be achieved. For any integer k,

$$(4k+2) - (4k+4) \ - (4k+5) \ - (4k+3) \ = 0$$

Thus

$$\begin{aligned} \cdots \quad \cdots \quad 2 - 4 \ - 5 \ - 3 \ - \cdots - (4k+2) \ - (4k+4) \ - (4k+5) \ - (4k+3) \ \cdots \\ -2010 \ - 2012 \ - 2013 \ - 2011 \ - 2014 \ - 1 \\ = \ 0 - 2014 \ - 1 \ = 2013 \end{aligned}$$

□

25. Answer: 1

Solution. Consider the complex number $\omega = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3}$. On one hand, using the binomial theorem one has

$$\begin{aligned} \mathrm{Re}(\omega^{2012}) &= \mathrm{Re}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{2012} \\ &= \mathrm{Re}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{2012} \\ &= \left(\frac{1}{2}\right)^{2012} - \binom{2012}{2}\left(\frac{1}{2}\right)^{2010}\left(\frac{3}{2^2}\right) + \frac{2012}{4}\left(\frac{1}{2}\right)^{2008}\left(\frac{3^2}{2^4}\right) \\ &\quad + \cdots + \left(\frac{3^{1006}}{2^{2012}}\right) \\ &= \frac{1}{2^{2012}}\left[1 - 3\binom{2012}{2} + 3^2\binom{2012}{4} + \cdots + 3^{1006}\binom{2012}{2012}\right] \end{aligned}$$

On the other hand, using the De Moivre's theorem one has

$$\mathrm{Re}(\omega^{2012}) = \mathrm{Re}\left(\cos\frac{2012\pi}{3} + i\sin\frac{2012\pi}{3}\right) = \cos\frac{2012\pi}{3} = \cos\frac{2\pi}{3} = -\frac{1}{2}$$

Thus,

$$\frac{1}{2^{2012}}\left[1 - 3\binom{2012}{2} + 3^2\binom{2012}{4} + \cdots - 3^{1004}\binom{2012}{2010} + 3^{1006}\binom{2012}{2012}\right] = -\frac{1}{2}$$

so that

$$\frac{-1}{2^{2011}}\left[1 - 3\binom{2012}{2} + 3^2\binom{2012}{4} + \cdots - 3^{1004}\binom{2012}{2010} + 3^{1006}\binom{2012}{2012}\right] = 1$$

□

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

(Open Section, Round 2)

Saturday, 30 June 2012 **0900-1300**

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1. The incircle with centre I of the triangle ABC touches the sides BC, CA and AB at D, E and F respectively. The line ID intersects the segment EF at K. Prove that A, K and M are collinear where M is the midpoint of BC.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that

$$(x + y)(f(x) - f(y)) = (x - y)f(x + y)$$

for all $x, y \in \mathbb{R}$.

3. For each $i = 1, 2, \ldots, N$, let a_i, b_i, c_i be integers such that at least one of them is odd. Show that one can find integers x, y, z such that $xa_i + yb_i + zc_i$ is odd for at least $4N/7$ different values of i.

4. Let p be an odd prime. Prove that

$$1^{p-2} + 2^{p-2} + 3^{p-2} + \cdots + \left(\frac{p-1}{2}\right)^{p-2} \equiv \frac{2 - 2^p}{p} \pmod{p}.$$

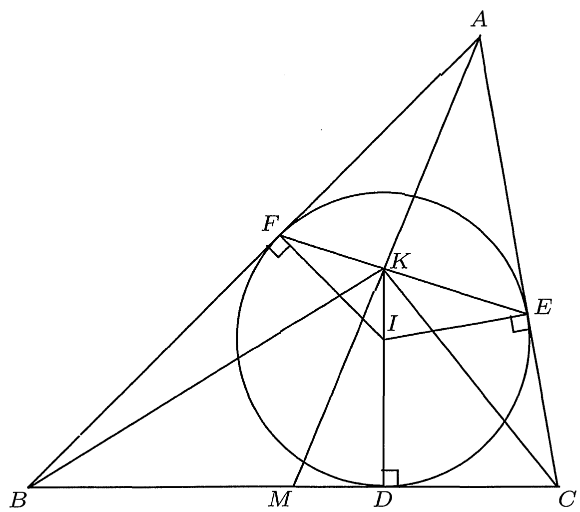
5. There are 2012 distinct points in the plane each of which is to be coloured using one of n colours so that the number of points of each colour are distinct. A set of n points is said to be multi-coloured if their colours are distinct. Determine n that maximizes the number of multi-coloured sets.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

(Open Section, Round 2 solutions)

1.



Let the line AK intersect BC at M. We shall prove that M is the midpoint of BC. Since $\angle FIK = \angle B$ and $\angle EIK = \angle C$, we have

$$\frac{FK}{EK} = \frac{\sin \angle FIK}{\sin \angle EIK} = \frac{\sin B}{\sin C}.$$

Also

$$\frac{FK}{\sin \angle FAK} = \frac{AF}{\sin \angle AKF} = \frac{AE}{\sin \angle AKE} = \frac{EK}{\sin \angle KAE}.$$

Therefore,

$$\frac{\sin \angle FAK}{\sin \angle KAE} = \frac{FK}{EK} = \frac{\sin B}{\sin C}.$$

Consequently,

$$\frac{BM}{CM} = \frac{BM}{AM} \cdot \frac{AM}{CM} = \frac{\sin \angle FAK}{\sin B} \cdot \frac{\sin C}{\sin \angle KAE} = 1,$$

so that $BM = CM$.

2. Suppose that f is a solution. Let

$$a = \frac{1}{2}(f(1) - f(-1)), \quad b = \frac{1}{2}(f(1) + f(-1))$$

and $g(x) = f(x) - ax - bx^2$. Then

$$(x+y)(g(x) - g(y)) = (x-y)g(x+y)$$

and $g(1) = g(-1) = 0$. Letting $y = 1$ and $y = -1$ above give

$$\begin{aligned}(x+1)g(x) &= (x-1)g(x+1) \\ xg(x+1) &= (x+2)g(x).\end{aligned}$$

Thus

$$x(x+1)g(x) = x(x-1)g(x+1) = (x-1)(x+2)g(x)$$

for all x. So $g(x) = 0$ for all x. Hence $f(x) = ax + bx^2$. We can check directly that any function of this form (for some $a, b \in \mathbb{R}$) satisfies the given equation.

3. Consider all the 7 triples (x, y, z), where x, y, z are either 0 or 1 but not all 0. For each i, at least one of the numbers a_i, b_i, c_i is odd. Thus among the 7 sums $xa_i + yb_i + zc_i$, 3 are even and 4 are odd. Hence there are altogether $4N$ odd sums. Thus there is choice of (x, y, z) for which at least $4N/7$ of the corresponding sums are odd. (You can think of a table where the rows are numbered $1, 2, \ldots, N$ and the columns correspond to the 7 choices of the triples (x, y, z). The 7 entries in row i are the 7 sums $xa_i + yb_i + zc_i$. Thus there are 4 odd numbers in each row, making a total of $4N$ odd sums in the table. Since there are 7 columns, one of the columns must contain at least $4N/7$ odd sums.)

4. First, for each $i = 1, 2, \cdots, \frac{p-1}{2}$,

$$\frac{2i}{p}\binom{p}{2i} = \frac{(p-1)(p-2)\cdots(p-(2i-1))}{(2i-1)!} \equiv \frac{(-1)(-2)\cdots(-(2i-1))}{(2i-1)!} \equiv -1 \pmod{p}.$$

Hence

$$\begin{aligned}\sum_{i=1}^{(p-1)/2} i^{p-2} &\equiv -\sum_{i=1}^{(p-1)/2} i^{p-2}\frac{2i}{p}\binom{p}{2i} \equiv -\frac{2}{p}\sum_{i=1}^{(p-1)/2} i^{p-1}\binom{p}{2i} \\ &\equiv -\frac{2}{p}\sum_{i=1}^{(p-1)/2}\binom{p}{2i} \pmod{p} \quad \text{(by Fermat's Little Theorem.)}\end{aligned}$$

The last summation counts the even-sized nonempty subsets of a p-element set, of which there are $2^{p-1} - 1$.

5. Let $m_1 < m_2 < \cdots < m_n$ be the number of points of each colour. We call $m_1, m_2, \ldots, m_n$ the colour distribution. Then $m_1 + \cdots + m_n = 2012$ and the number of multi-coloured sets is $M = m_1 m_2 \ldots m_n$. We have the following observations.

(i) $m_1 > 1$. For if $m_1 = 1$, then $m_1 m_2 \ldots m_n < m_2 m_3 \ldots m_{n-1}(1 + m_n)$. This means if we use $n - 1$ colours with colour distribution $m_2, m_3, \ldots, m_{n-1}, (1 + m_n)$, we obtain a larger M.

(ii) $m_{i+1} - m_i \leq 2$ for all i. For if there exists k with $m_{k+1} - m_k \geq 3$, then the colour distribution with m_k, m_{k+1} replaced by $m_k + 1, m_{k+1} - 1$ yields a larger M.

(iii) $m_{i+1} - m_i = 2$ for at most one i. For if there exist $i < j$ with $m_{i+1} - m_i = m_{j+1} - m_j = 2$, the colour distribution with m_i, m_{j+1} replaced by $m_i + 1, m_{j+1} - 1$ yields a larger M.

(iv) $m_{i+1} - m_i = 2$ for exactly one i. For if $m_{i+1} - m_i = 1$ for all i, then $m_1 + \cdots + m_n = nm_1 + \frac{n(n-1)}{2} = 2012 = 4 \cdot 503$. Thus $n(2m_1 - 1 + n) = 8 \cdot 503$. Since 503 is prime, the parity of n and $2m_1 - 1 + n$ are opposite and $2m_1 - 1 + n > n$, we have $n = 8$ and $m_1 = 248$. The colour distribution with m_1 replaced by two numbers $2, 246$ (using $n + 1$ colours) yields a larger M.

(v) $m_1 = 2$. If $m_n - m_{n-1} = 2$, then from (iv), we have $m_1 + \cdots + m_n = nm_1 + \frac{n(n-1)}{2} + 1 = 2012$. Thus $n(2m_1 - 1 + n) = 2 \cdot 2011$. Since 2011 is prime, we get $n = 2$ and $m_1 = 1005$ which will lead to a contradiction as in (iv). Thus $m_n - m_{n-1} = 1$. $m_{i+1} - m_i = 2$ for some $1 \leq i \leq n - 2$. Suppose $m_1 \geq 3$. Let $m' = m_{i+2} - 2$. Then $m_i < m' < m_{i+1}$ with replacing m_{i+2} by $2, m'$ yields a larger M. Thus $m_1 = 2$.

From the above analysis, with n colours, we see that the colour distribution 2, 3, $\ldots$, $i - 1$, $i + 1$, $i + 2$, $\ldots$, $n + 1$, $n + 2$, with $3 \leq i \leq n$, yields the maximum M. Now we have $\sum m_i = \frac{1}{2}(n + 1)(n + 4) - i = 2012$. Thus $n^2 + 5n - 4020 = 2i$, $3 \leq i \leq n$, i.e., $n^2 + 5n \geq 4026$ and $n^2 + 3n \leq 4020$. Thus $n = 61$ and $i = 3$. Thus the maximum is achieved when $n = 61$ with the colour distribution $2, 4, 5, 6, \ldots, 63$.

Singapore Mathematical Olympiad 2012 Errata

Junior Section 2012

Page 2: Q5, line 3: DE/AE. (Slash in DE/AE missing.)

Page 3: Q8, line 1: $\{x\} = x - \lfloor x \rfloor$. (Braces in $\{x\}$ missing.)

Page 4: Q9, line 2: $y = \dfrac{|x - |x||}{x}$. (Absolute value sign misprinted.)

Page 9: Q1, line 3: $4[(b-1/2)^2 + 3/4] \geq 3$. (Slash in $(b-1/2)^2$ and $3/4$ missing.)

Q2, line 3: $5 \mid (n^2+1)$, $2 \mid n^{2010}$, $10 \mid (n^2+1)$. ($\mid$ missing in $5 \mid (n^2+1)$, $2 \mid n^{2010}$ and $10 \mid (n^2+1)$.)

Q2, line 4: $2 \mid (n^2+1)$, $5 \mid n^{2010}$. ($\mid$ missing in $2 \mid (n^2+1)$ and $5 \mid n^{2010}$.)

Q4, line 2: $\dfrac{2(1-1/\sqrt{2})}{1}$. (Slash in $1/\sqrt{2}$ missing.)

Page 11: Q9, Answer (D). (Answer misprinted as (C).)

Q9, line 1: $y = \dfrac{|x-|x||}{x} = \dfrac{|x-x|}{x} = \dfrac{|0|}{x} = 0$. (Absolute sign misprinted.)

Q9, line 2: $y = \dfrac{|x-|x||}{x} = \dfrac{|x-(-x)|}{x} = \dfrac{|2x|}{x} = \dfrac{-2x}{x} = -2$. (Absolute sign misprinted.)

Q12, line 1: $(x^2-1) \mid 120$. ($\mid$ misprinted in $(x^2-1) \mid 120$.)

Page 12: Q16, line 3: $(mn-217) \mid 8$. ($\mid$ missing in $(mn-217) \mid 8$.)

Page 13: Q17, line 1: $x = \lfloor x \rfloor + \{x\}$, $100 \leq (\lfloor x \rfloor + \{x\})^2 - \lfloor x \rfloor^2 = 2\lfloor x \rfloor\{x\} + \{x\}^2 < 2\lfloor x \rfloor + 1$. (Braces in $\{x\}$ missing.)

Q20, line 5: $(n-1) \mid 9$. ($\mid$ misprinted in $(n-1) \mid 9$.)

Page 14: Q24, line 1: $(a+b) \mid 9a$. ($\mid$ missing in $(a+b) \mid 9a$.)

Q24, line 2: $(a+b) \mid a$. ($\mid$ missing in $(a+b) \mid a$.)

Q24, line 3: $3 \mid (a+b)$, $(a+b) \mid 3a$. ($\mid$ missing in $3 \mid (a+b)$ and $(a+b) \mid 3a$.)

Page 15: Q27, line 1: $\lambda = EG = 3/4$. (Slash in $3/4$ missing.)

Singapore Mathematical Olympiad 2012 Errata

Senior Section 2012

Page 22: Q3, line 1: $T = \{2, 3, 5, 6, 7, 8, 10, 11, \ldots\}$. (Braces and dots missing.)

Page 24: Q13, line 1: $S = \{1, 2, \ldots, 10\}$. (Braces, commas and dots missing.)

Page 25: Q22, line 1: $\{1, 2, 3, 4, 5, 6\}$. (Braces missing.)

Page 25: Q25, line 1: $S = \{1, 2, 3, \ldots, 19, 20\}$. (Braces missing.)

Page 26: Q29, line 2: $f(x, y, z)$. (Commas missing.)

Page 26: Q30, line 2: $k = 0, 1, 2, \ldots, 34$. (Commas and dots missing.)

Page 29: Q8, line 5: $[2(k-1)\pi, (2k-1)\pi]$, ..., $((2k-1)\pi, 2k\pi)$. (Commas missing.)

Page 29: Q10, lines 2,3 and 4: $1000 \mid 12^n(12^{m-n}-1)$, $8 \mid 12^n$ and $125 \mid 12^{m-n}-1$. ($|$ missing in the three expressions.)

Page 30: Q12, lines 5 and 8: $3 \mid y$ or $3 \mid y+2$ and $3 \mid y+2$. ($|$ missing in the two expressions.)

Page 30: Q13, line 4: $|\mathcal{F}| \leq \dfrac{2^{10}}{2} = 512$. ($|\ \ |$ missing from $|\mathcal{F}|$.)

Page 33: Q25, lines 2 and 3: $S_0 = \{3, 6, \ldots, 18\}$, $S_1 = \{1, 4, \ldots, 19\}$ and $S_2 = \{2, 5, \ldots, 20\}$. (Braces missing in the three expressions.)

Page 34: Q30, line 3: $k = 0, 1, 2, \ldots, n$. (Commas and dots missing.)

Page 35: Q32, lines 6 and 7: Thus $0 \leq e \leq 16/5$ and $a = b = c = d = 6/5$, we have $e = 16/5$. (Slash in 16/5 and 6/5 missing.)

Page 35: Q33, lines 3 to 9: $F(a, b, c)$, $F(a, b, 9)$, $F(a, 8, 9)$ and $F(1, 8, 9) = 10.5$. (Commas in all these expressions and decimal point in the last expression missing.)

Page 36: Q35, line 3: $(0, 1, 1, 2, \ldots)$ and $(n + f(n))_{n=1}^{\infty} = (2, 3, 5, 6, \ldots)$. (Commas and dots missing.)

Singapore Mathematical Olympiad 2012 Errata

Senior Section (Round 2) 2012

Page 37: Q4, lines 1,3 and 4: $a_1, a_2, \ldots, a_n, a_{n+1}$, $k = 1, 2, \ldots, n$ and $k = 0, 1, \ldots, n+1$. (Dots missing.)

Singapore Mathematical Olympiad 2012 Errata

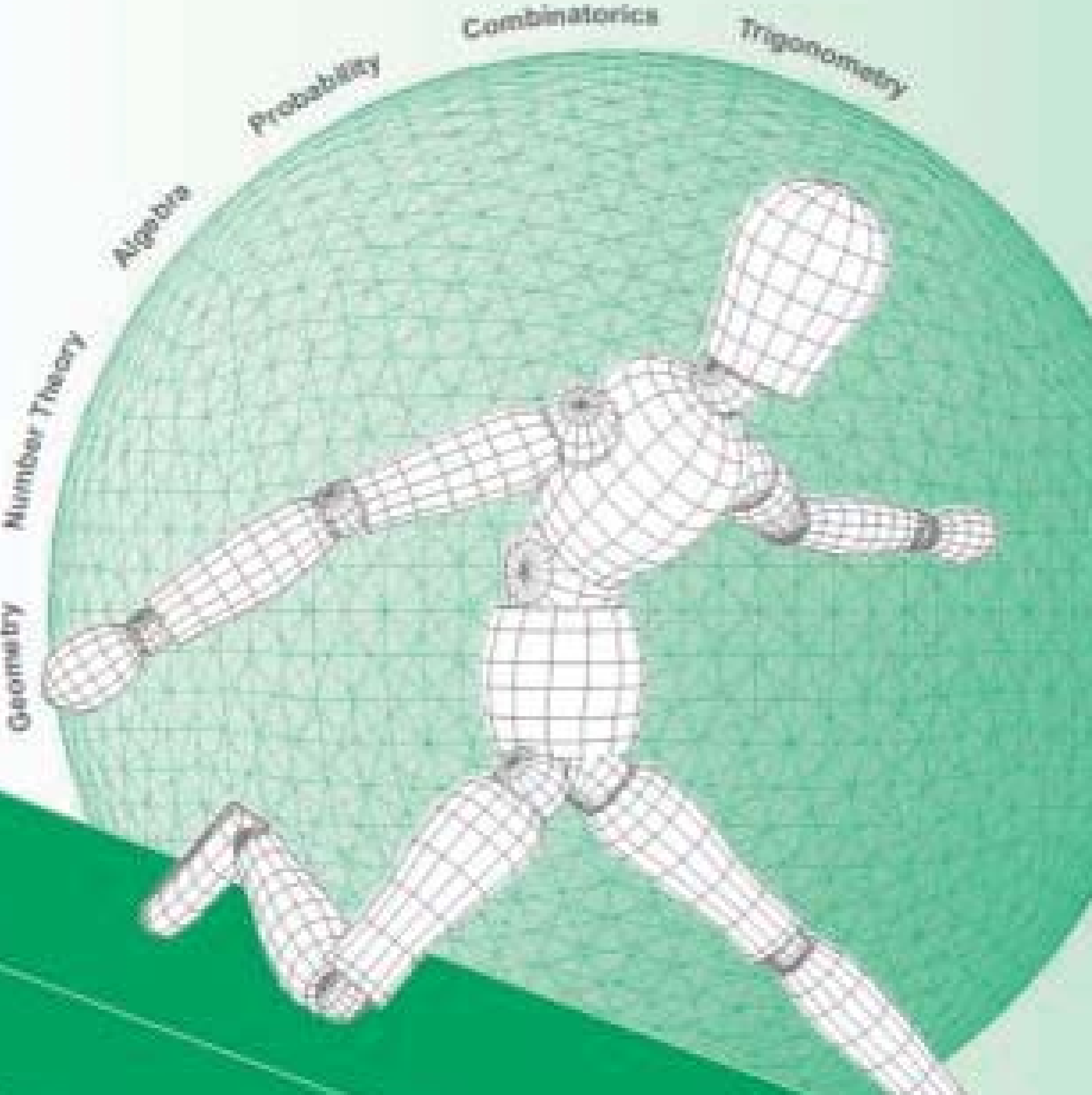
Open Section 2012

Page 42: Q16, line 2: $x_1, x_2, x_3, \ldots, x_k$. (Commas missing.)

Page 43: Q24, line 2: $|\cdots|||x_1 - x_2| - x_3| - x_4|\cdots - x_{2014}|$. (Absolute value sign misprinted.)

Page 43: Q24, line 3: $\{1, 2, 3, 4, \cdots, 2014\}$. (Braces misprinted.)

SINGAPORE MATHEMATICAL OLYMPIADS
2013



Tay Tiong Seng
Toh Pee Choon
Toh Tin Lam
Wang Fei

Published by Singapore Mathematical Society

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

Junior Section (First Round)

Tuesday, 4 June 2013 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x. *For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. If $a = 8^{53}$, $b = 16^{41}$ and $c = 64^{27}$, then which of the following inequalities is true?

   (A) $a > b > c$ (B) $c > b > a$ (C) $b > a > c$ (D) $b > c > a$ (E) $c > a > b$

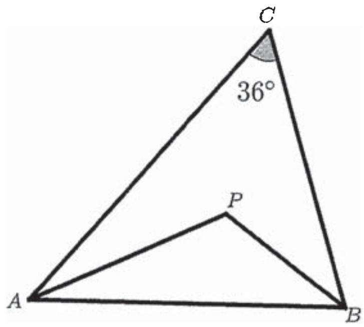
2. If a, b, c are real numbers such that $|a-b| = 1$, $|b-c| = 1$, $|c-a| = 2$ and $abc = 60$, find the value of $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}$.

   (A) $\frac{1}{30}$ (B) $\frac{1}{20}$ (C) $\frac{1}{10}$ (D) $\frac{1}{4}$ (E) None of the above

3. If x is a complex number satisfying $x^2 + x + 1 = 0$, what is the value of $x^{49} + x^{50} + x^{51} + x^{52} + x^{53}$?

   (A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

4. In $\triangle ABC$, $\angle ACB = 36°$ and the interior angle bisectors of $\angle CAB$ and $\angle ABC$ intersect at P. Find $\angle APB$.



   (A) 72° (B) 108° (C) 126° (D) 136° (E) None of the above

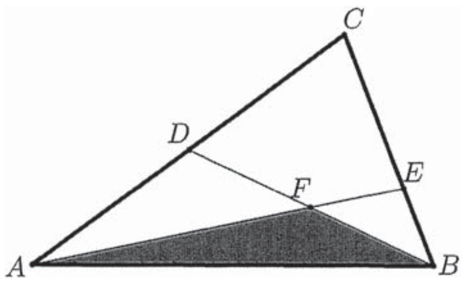
5. Find the number of integer pairs x, y such that $xy - 3x + 5y = 0$.

   (A) 1 (B) 2 (C) 4 (D) 8 (E) 16

6. Five young ladies were seated around a circular table. Miss Ong was sitting between Miss Lim and Miss Mak. Ellie was sitting between Cindy and Miss Nai. Miss Lim was between Ellie and Amy. Lastly, Beatrice was seated with Miss Poh on her left and Miss Mak on her right. What is Daisy's surname?

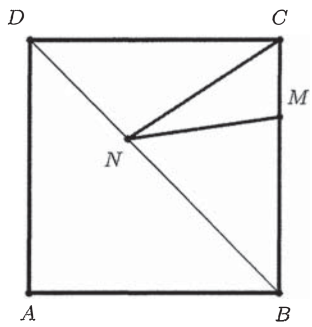
   (A) Lim (B) Mak (C) Nai (D) Ong (E) Poh

7. Given that ABC is a triangle with D being the midpoint of AC and E a point on CB such that $CE = 2EB$. If AE and BD intersect at point F and the area of $\triangle AFB = 1$ unit, find the area of $\triangle ABC$.



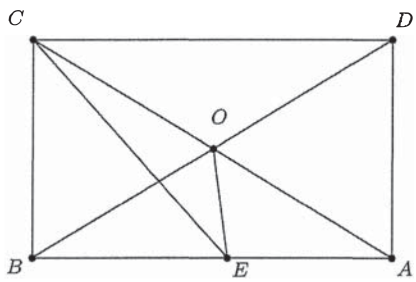
(A) 3 (B) $\frac{10}{3}$ (C) $\frac{11}{3}$ (D) 4 (E) 5

8. $ABCD$ is a square with sides 8 cm. M is a point on CB such that $CM = 2$ cm. If N is a variable point on the diagonal DB, find the least value of $CN + MN$.



(A) 8 (B) $6\sqrt{2}$ (C) 10 (D) $8\sqrt{2}$ (E) 12

9. $ABCD$ is a rectangle whose diagonals intersect at point O. E is a point on AB such that CE bisects $\angle BCD$. If $\angle ACE = 15°$, find $\angle BOE$.



(A) 60° (B) 65° (C) 70° (D) 75° (E) 80°

10. Let S be the smallest positive multiple of 15, that comprises exactly $3k$ digits with k '0's, k '3's and k '8's. Find the remainder when S is divided by 11.

(A) 0 (B) 3 (C) 5 (D) 6 (E) 8

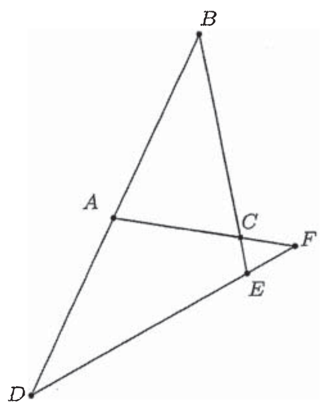
Short Questions

11. Find the value of $\sqrt{9999^2 + 19999}$.

12. If the graphs of $y = x^2 + 2ax + 6b$ and $y = x^2 + 2bx + 6a$ intersect at only one point in the xy-plane, what is the x-coordinate of the point of intersection?

13. Find the number of multiples of 11 in the sequence $99, 100, 101, 102, \cdots, 20130$.

14. In the figure below, BAD, BCE, ACF and DEF are straight lines. It is given that $BA = BC$, $AD = AF$, $EB = ED$. If $\angle BED = x°$, find the value of x.

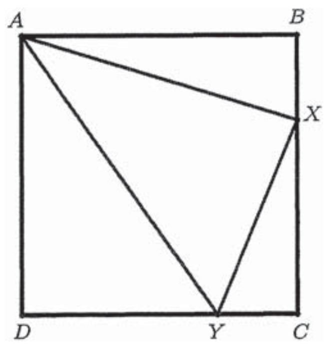


15. If $a = 1.69$, $b = 1.73$ and $c = 0.48$, find the value of

$$\frac{1}{a^2 - ac - ab + bc} + \frac{2}{b^2 - ab - bc + ac} + \frac{1}{c^2 - ac - bc + ab}.$$

16. Suppose that x_1 and x_2 are the two roots of the equation $(x - 2)^2 = 3(x + 5)$. What is the value of the expression $x_1x_2 + x_1^2 + x_2^2$?

17. Let $ABCD$ be a square and X and Y be points such that the lengths of XY, AX and AY are 6, 8 and 10 respectively. The area of $ABCD$ can be expressed as $\frac{m}{n}$ units where m and n are positive integers without common factors. Find the value of $m + n$.



18. Let x and y be real numbers satisfying the inequality

$$5x^2 + y^2 - 4xy + 24 \leq 10x - 1.$$

Find the value of $x^2 + y^2$.

19. A painting job can be completed by Team A alone in 2.5 hours or by Team B alone in 75 minutes. On one occasion, after Team A had completed a fraction $\frac{m}{n}$ of the job, Team B took over immediately. The whole painting job was completed in 1.5 hours. If m and n are positive integers with no common factors, find the value of $m + n$.

20. Let a, b and c be real numbers such that $\dfrac{ab}{a+b} = \dfrac{1}{3}$, $\dfrac{bc}{b+c} = \dfrac{1}{4}$ and $\dfrac{ca}{c+a} = \dfrac{1}{5}$. Find the value of $\dfrac{24abc}{ab+bc+ca}$.

21. Let x_1 and x_2 be two real numbers that satisfy $x_1 x_2 = 2013$. What is the minimum value of $(x_1 + x_2)^2$?

22. Find the value of $\sqrt{45 - \sqrt{2000}} + \sqrt{45 + \sqrt{2000}}$.

23. Find the smallest positive integer k such that $(k - 10)^{4026} \geq 2013^{2013}$.

24. Let a and b be two real numbers. If the equation $ax + (b - 3) = (5a - 1)x + 3b$ has more than one solution, what is the value of $100a + 4b$?

25. Let $S = \{1, 2, 3, \ldots, 48, 49\}$. What is the maximum value of n such that it is possible to select n numbers from S and arrange them in a circle in such a way that the product of any two adjacent numbers in the circle is less than 100?

26. Given any 4-digit positive integer x not ending in '0', we can reverse the digits to obtain another 4-digit integer y. For example if x is 1234 then y is 4321. How many possible 4-digit integers x are there if $y - x = 3177$?

27. Find the least positive integer n such that $2^8 + 2^{11} + 2^n$ is a perfect square.

28. How many 4-digit positive multiples of 4 can be formed from the digits $0, 1, 2, 3, 4, 5, 6$ such that each digit appears without repetition?

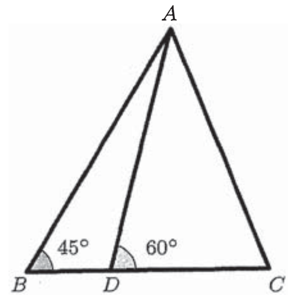
29. Let m and n be two positive integers that satisfy

$$\frac{m}{n} = \frac{1}{10 \times 12} + \frac{1}{12 \times 14} + \frac{1}{14 \times 16} + \cdots + \frac{1}{2012 \times 2014}.$$

Find the smallest possible value of $m + n$.

30. Find the units digit of $2013^1 + 2013^2 + 2013^3 + \cdots + 2013^{2013}$.

31. In $\triangle ABC$, $DC = 2BD$, $\angle ABC = 45°$ and $\angle ADC = 60°$. Find $\angle ACB$ in degrees.



32. If a and b are positive integers such that $a^2 + 2ab - 3b^2 - 41 = 0$, find the value of $a^2 + b^2$.

33. Evaluate the following sum

$$\left\lfloor \frac{1}{1} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor + \left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{3}{3} \right\rfloor + \left\lfloor \frac{1}{4} \right\rfloor + \left\lfloor \frac{2}{4} \right\rfloor + \left\lfloor \frac{3}{4} \right\rfloor + \left\lfloor \frac{4}{4} \right\rfloor + \left\lfloor \frac{1}{5} \right\rfloor + \cdots,$$

up to the 2013^{th} term.

34. What is the smallest possible integer value of n such that the following statement is always true?

In any group of $2n - 10$ *persons, there are always at least* 10 *persons who have the same birthdays.*

(For this question, you may assume that there are exactly 365 different possible birthdays.)

35. What is the smallest positive integer n, where $n \neq 11$, such that the highest common factor of $n - 11$ and $3n + 20$ is greater than 1?

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

Junior Section (First Round Solutions)

Multiple Choice Questions

1. Answer: (D)

   First note that $c = (8^2)^{27} = 8^{54}$, so we see that $c > a$. Next, $b = (4^2)^{41} = 4^{82}$ and $c = (4^3)^{27} = 4^{81}$. Therefore we have $b > c$. Consequently $b > c > a$.

2. Answer: (B)

$$\begin{aligned}\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c} &= \frac{a^2+b^2+c^2-bc-ca-ab}{abc}\\ &= \frac{(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca+a^2)}{2abc}\\ &= \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2abc}\\ &= \frac{1^2+1^2+2^2}{120}=\frac{1}{20}.\end{aligned}$$

3. Answer: (A)

   Note that $x^2+x+1 = \dfrac{x^3-1}{x-1}$, so $x^2+x+1=0$ implies that $x^3=1$ and $x \neq 1$. Now

$$\begin{aligned}x^{49}+x^{50}+x^{51}+x^{52}+x^{53} &= x^{49}(1+x+x^2)+x^{51}(x+x^2)\\ &= x^{49}\times 0+(x^3)^{17}(-1)\\ &= 1^{17}\times(-1)=-1.\end{aligned}$$

4. Answer: (B)

   Let $\angle CAB = x$ and $\angle ABC = y$. Then $x+y = 180^\circ - 36^\circ = 144^\circ$.
   Now $\angle APB = 180^\circ - \frac{x+y}{2} = 108^\circ$.

5. Answer: (D)

   $xy - 3x + 5y = 0$ is equivalent to $(x+5)(y-3) = -15$.
   If $x+5 = a$ and $y-3 = b$, then there are eight distinct pairs of integers a, b (counting signs) such that $ab = -15$.

6. Answer: (B)

Beatrice, being between Miss Poh and Miss Mak cannot be Miss Ong who was between Miss Lim and Miss Mak. This means that we have in order from the left, Miss Poh, Beatrice, Miss Mak, Miss Ong and Miss Lim. So Beatrice must be Miss Nai. Since Ellie was beside Miss Nai and also besides Miss Lim, she must be Miss Poh. This implies Cindy is Miss Lim and Amy was Miss Ong leaving Daisy as Miss Mak.

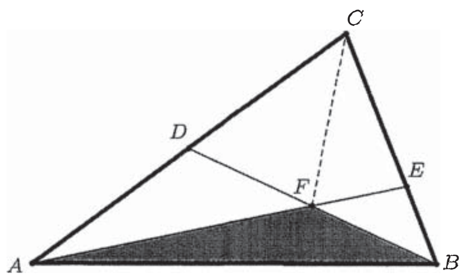
7. Answer: (D)

Construct a line joining C and F. Then using $[XYZ]$ to denote the area of $\triangle XYZ$, we know that $[ADF] = [DCF] = x$ and if $[BFE] = z$, then $[FCE] = 2z$.

Furthermore, we have $[ADB] = [DCB]$ i.e. $x + 1 = x + 3z$, so $z = \frac{1}{3}$.

Also, $2 \times [AEB] = [ACE]$ i.e. $2 + 2z = 2x + 2z$, so $x = 1$.

In conclusion, $[ABC] = 1 + 2x + 3z = 4$ units.



8. Answer: (C)

Join A to N. By symmetry, $AN + NM = MN + CN$, and the least value occurs when ANM is a straight line. Thus the least value is

$$\sqrt{AB^2 + BM^2} = \sqrt{8^2 + 6^2} = 10.$$

9. Answer: (D)

Since CE bisects $\angle BCD$, $\angle BCE = 45°$. Thus $\angle CEB = 45°$ also and $\triangle CBE$ is isosceles. Therefore $BC = BE$.

Now $\angle BCO = 45° + 15° = 60°$. As $CO = BO$, we conclude that $\triangle COB$ is equilateral. Thus $BC = BO = BE$ giving us an isosceles triangle OBE. Since $\angle OBE = 30°$, thus $\angle BOE = 75°$.

10. Answer: (D)

S being a multiple of 5 and 3 must end with '0' and has the sum of digits divisible by 3. Since $3 + 8 = 11$, the smallest positive k such that $k \times 11$ is divisible by 3 is 3. Thus $S = 300338880$ and the remainder is

$$0 - 8 + 8 - 8 + 3 - 3 + 0 - 0 + 3 = -5 \equiv 6 \pmod{11}.$$

Short Questions

11. Answer: 10000

$\sqrt{9999^2+19999} = \sqrt{9999^2+2\times 9999+1} = \sqrt{(9999+1)^2} = 10000.$

12. Answer: 3

Let (α, β) be the point of intersection of the two graphs. Then

$$\beta = \alpha^2 + 2a\alpha + 6b = \alpha^2 + 2b\alpha + 6a.$$

It follows that $2(a-b)\alpha = 6(a-b)$. Since the two graphs intersect at only one point, we see that $a-b \neq 0$ (otherwise the two graphs coincide and would have infinitely many points of intersection). Consequently $2\alpha = 6$, and hence $\alpha = 3$.

13. Answer: 1822

The the number of multiples of 11 in the sequence $1, 2, ..., n$ is equal to $\lfloor \frac{n}{11} \rfloor$. Thus the answer to this question is $\left\lfloor \dfrac{20130}{11} \right\rfloor - \left\lfloor \dfrac{98}{11} \right\rfloor = 1830 - 8 = 1822.$

14. Answer: 108

Let $\angle ABC = \alpha$ and $\angle BAC = \beta$. Since $BA = BC$, we have $\angle BCA = \angle BAC = \beta$. As $EB = ED$, it follows that $\angle EDB = \angle EBD = \angle ABC = \alpha$. Then $\angle AFD = \angle ADF = \angle EDB = \alpha$ since $AD = AF$. Note that $\angle DAF = 180^\circ - \beta$. In $\triangle ABC$, we have $\alpha + 2\beta = 180^\circ$; and in $\triangle ADF$, we have $2\alpha + 180^\circ - \beta = 180^\circ$. From the two equations, we obtain $\alpha = 36^\circ$. By considering $\triangle BDE$, we obtain $x = 180^\circ - 2\alpha = 108^\circ$.

15. Answer: 20

$$\begin{aligned}
&\frac{1}{a^2-ac-ab+bc} + \frac{2}{b^2-ab-bc+ac} + \frac{1}{c^2-ac-bc+ab} \\
&= \frac{1}{(a-b)(a-c)} + \frac{2}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)} \\
&= \frac{c-b-2(c-a)-(a-b)}{(a-b)(b-c)(c-a)} \\
&= \frac{-1}{(a-b)(b-c)} = 20.
\end{aligned}$$

16. Answer: 60

The equation $(x-2)^2 = 3(x+5)$ is equivalent to $x^2 - 7x - 11 = 0$. Thus $x_1 + x_2 = 7$ and $x_1x_2 = -11$. So

$$x_1x_2 + x_1^2 + x_2^2 = (x_1+x_2)^2 - x_1x_2 = 7^2 - (-11) = 60.$$

17. Answer: 1041

Let the length of the side be s. Observe that since $6^2+8^2=10^2$ so $\angle AXY=90°$. This allows us to see that $\triangle ABX$ is similar to $\triangle XCY$. Thus $\dfrac{AX}{XY}=\dfrac{AB}{XC}$, i.e. $\dfrac{8}{6}=\dfrac{s}{s-BX}$. Solving this equation gives $s=4BX$ and we can then compute that

$$8^2=AB^2+BX^2=16BX^2+BX^2.$$

So $BX=\dfrac{8}{\sqrt{17}}$ and $s^2=16\times\dfrac{64}{17}=\dfrac{1024}{17}$. Thus $m+n=1041$.

18. Answer: 125

The inequality is equivalent to

$$(x-5)^2+(2x-y)^2\le 0.$$

Thus we must have $(x-5)=0$ and $(2x-y)=0$, hence $x^2+y^2=5^2+10^2=125$.

19. Answer: 6

Suppose Team B spent t minutes on the job. Then

$$\frac{t}{75}+\frac{90-t}{150}=1.$$

Thus $t=60$ minutes and so Team A completed $\frac{30}{150}=\frac{1}{5}$ of the job. So $m+n=6$.

20. Answer: 4

Taking reciprocals, we find that $\frac{1}{a}+\frac{1}{b}=3$, $\frac{1}{b}+\frac{1}{c}=4$ and $\frac{1}{a}+\frac{1}{c}=5$. Summing the three equations, we get

$$12=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=2\times\frac{ab+bc+ca}{abc}.$$

Hence $\dfrac{24abc}{ab+bc+ca}=4$.

21. Answer: 8052

$$(x_1+x_2)^2=(x_1-x_2)^2+4x_1x_2\ge 0+4\times 2013=8052.$$

If $x_1=x_2=\sqrt{2013}$, then $(x_1+x_2)^2=8052$.

22. Answer: 10

Let $x_1=\sqrt{45-\sqrt{2000}}$ and $x_2=\sqrt{45+\sqrt{2000}}$. Then $x_1^2+x_2^2=90$ and

$$x_1x_2=\sqrt{(45-\sqrt{2000})(45+\sqrt{2000})}=\sqrt{45^2-2000}=\sqrt{25}=5.$$

Thus

$$(x_1+x_2)^2=x_1^2+x_2^2+2x_1x_2=100.$$

As both x_1 and x_2 are positive, we have $x_1+x_2=10$.

23. Answer: 55

$(k-10)^{4026} = ((k-10)^2)^{2013} \geq 2013^{2013}$ is equivalent to to $(k-10)^2 \geq 2013$. As $k-10$ is an integer and $44^2 < 2013 < 45^2$, the minimum value of $k-10$ is 45, and thus the minimum value of k is 55.

24. Answer: 19

Rearranging the terms of the equation, we obtain

$$(1-4a)x = 2b+3.$$

Since the equation has more than one solution (i.e., infinitely many solutions), we must have $1-4a=0$ and $2b+3=0$. Therefore $a=\frac{1}{4}$ and $b=-\frac{3}{2}$. Consequently, $100a+4b=19$.

25. Answer: 18

First note that the product of any two different 2-digit numbers is greater than 100. Thus if a 2-digit number is chosen, then the two numbers adjacent to it in the circle must be single-digit numbers. Note that at most nine single-digit numbers can be chosen from S, and no matter how these nine numbers $1, 2, \ldots, 9$ are arranged in the circle, there is at most one 2-digit number in between them. Hence it follows that $n \leq 18$. Now the following arrangement

$$1, 49, 2, 33, 3, 24, 4, 19, 5, 16, 6, 14, 7, 12, 8, 11, 9, 10, 1$$

shows that $n \geq 18$. Consequently we conclude that the maximum value of n is 18.

26. Answer: 48

Let $x = \overline{abcd}$ and $y = \overline{dcba}$ where $a, d \neq 0$. Then

$$\begin{aligned} y - x &= 1000 \times d - d + 100 \times c - 10 \times c + 10 \times b - 100 \times b + a - 1000 \times a \\ &= 999(d-a) + 90(c-b) = 9\left(111(d-a) + 10(c-b)\right). \end{aligned}$$

So we have $111(d-a)+10(c-b) = 353$. Consider the remainder modulo 10, we obtain $d-a=3$, which implies that $c-b=2$. Thus the values of a and b determines the values of d and c respectively.

a can take on any value from 1 to 6, and b can take any value from 0 to 7, giving $6 \times 8 = 48$ choices.

27. Answer: 12

Let $2^8 + 2^{11} + 2^n = m^2$ and so

$$2^n = m^2 - 2^8(1+8) = (m-48)(m+48).$$

If we let $2^k = m+48$, then $2^{n-k} = m-48$ and we have

$$2^k - 2^{n-k} = 2^{n-k}(2^{2k-n}-1) = 96 = 2^5 \times 3.$$

This means that $n-k=5$ and $2k-n=2$, giving us $n=12$.

28. Answer: 208

Note that a positive integer k is a multiple of 4 if and only if the number formed by the last two digits of k (in the same order) is a multiple of 4. There are 12 possible multiples of 4 that can be formed from the digits $0, 1, 2, 3, 4, 5, 6$ without repetition, namely

$$20, 40, 60, 12, 32, 52, 04, 24, 64, 16, 36, 56.$$

If 0 appears in the last two digits, there are 5 choices for the first digit and 4 choices for the second digit. But if 0 does not appear, there are 4 choices for the first digit and also 4 choices for the second digit. Total number is

$$4 \times 5 \times 4 + 8 \times 4 \times 4 = 208.$$

29. Answer: 10571

$$\begin{aligned} \frac{m}{n} = \frac{1}{4} \sum_{k=5}^{1006} \frac{1}{k(k+1)} &= \frac{1}{4} \sum_{k=5}^{1006} \frac{1}{k} - \frac{1}{k+1} \\ &= \frac{1}{4}\left(\frac{1}{5} - \frac{1}{1007}\right) \\ &= \frac{501}{10070}. \end{aligned}$$

Since $\gcd(501, 10070) = 1$, we have $m + n = 10571$.

30. Answer: 3

Note that the units digit of $2013^1 + 2013^2 + 2013^3 + \cdots + 2013^{2013}$ is equal to the units digit of the following number

$$3^1 + 3^2 + 3^3 + \cdots + 3^{2013}.$$

Since $3^2 = 9, 3^3 = 27, 3^4 = 81$, the units digits of the sequence of $3^1, 3^2, 3^3, 3^4, \cdots, 3^{2013}$ are

$$\underbrace{3, 9, 7, 1, 3, 9, 7, 1, \cdots, 3, 9, 7, 1}_{\text{2012 numbers}}, 3.$$

Furthermore the sum $3 + 9 + 7 + 1$ does not contribute to the units digit, so the answer is 3.

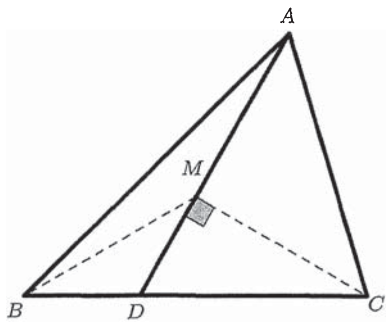
31. Answer: 75

Construct a point M on AD so that CM is perpendicular to AD. Join B and M.

Since $\angle ADC = 60°$, $\angle MCD = 30°$. As $\sin 30° = \frac{1}{2}$, so $2MD = DC$. This means that $BD = MD$ and $\triangle MDB$ is isosceles. It follows that $\angle MBD = 30°$ and $\angle ABM = 15°$.

We further observe that $\triangle MBC$ is also isosceles and thus $MB = MC$.

Now $\angle BAM = \angle BMD - \angle ABM = 15°$, giving us yet another isosceles triangle $\triangle BAM$. We now have $MC = MB = MA$, so $\triangle AMC$ is also isosceles. This allows us to calculate $\angle ACM = 45°$ and finally $\angle ACB = 30° + 45° = 75°$.



32. Answer: 221

We have $a^2+2ab-3b^2 = (a-b)(a+3b) = 41$. Since 41 is a prime number, and $a-b < a+3b$, we have $a - b = 1$ and $a + 3b = 41$. Solving the simultaneous equations gives $a = 11$ and $b = 10$. Hence $a^2 + b^2 = 221$.

33. Answer: 62

We first note that for $1 \leq r < k$, $\lfloor \frac{r}{k} \rfloor = 0$ and $\lfloor \frac{k}{k} \rfloor = 1$. The total number of terms up to $\lfloor \frac{N}{N} \rfloor$ is given by $\frac{1}{2}N(N+1)$, and we have the inequality

$$\frac{62(63)}{2} = 1953 < 2013 < 2016 = \frac{63(64)}{2}.$$

So the 2013^{th} term is $\lfloor \frac{60}{63} \rfloor$, and the sum up to this term is just 62.

34. Answer: 1648

By the pigeonhole principle in any group of $365 \times 9 + 1 = 3286$ persons, there must be at least 10 persons who share the same birthday.

Hence solving $2n - 10 \geq 3286$ gives $n \geq 1648$. Thus the smallest possible n is 1648 since $2 \times 1647 - 10 = 3284 < 365 \times 9$, and it is possible for each of the 365 different birthdays to be shared by at most 9 persons.

35. Answer: 64

Let $d > 1$ be the highest common factor of $n - 11$ and $3n + 20$. Then $d \mid (n - 11)$ and $d \mid (3n + 20)$. Thus $d \mid [3n + 20 - 3(n - 11)]$, i.e., $d \mid 53$. Since 53 is a prime and $d > 1$, it follows that $d = 53$. Therefore $n - 11 = 53k$, where k is a positive integer, so $n = 53k + 11$. Note that for any k, $3n+20$ is a multiple of 53 since $3n+20 = 3(53k+11)+20 = 53(3k+1)$. Hence $n = 64$ (when $k = 1$) is the smallest positive integer such that $\mathrm{HCF}(n-11, 3n+20) > 1$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Junior Section, Round 2)

Saturday, 29 June 2013 **0930-1230**

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1. Let $a < b < c < d < e$ be real numbers. Among the 10 sums of the pairs of these numbers, the least three are 32, 36 and 37 while the largest two are 48 and 51. Find all possible values of e.

2. In the triangle ABC, points D, E, F are on the sides BC, CA and AB respectively such that FE is parallel to BC and DF is parallel to CA. Let P be the intersection of BE and DF, and Q the intersection of FE and AD. Prove that PQ is parallel to AB.

3. Find all primes that can be written both as a sum of two primes and as a difference of two primes.

4. Let a and b be positive integers with $a > b > 2$. Prove that $\frac{2^a+1}{2^b-1}$ is not an integer.

5. Six musicians gathered at a chamber music festival. At each scheduled concert some of the musicians played while the others listened as members of the audience. What is the least number of such concerts which would need to be scheduled so that for every two musicians each must play for the other in some concert?

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Junior Section, Round 2 solutions)

1. We have 37 is either $a+d$ or $b+c$ and

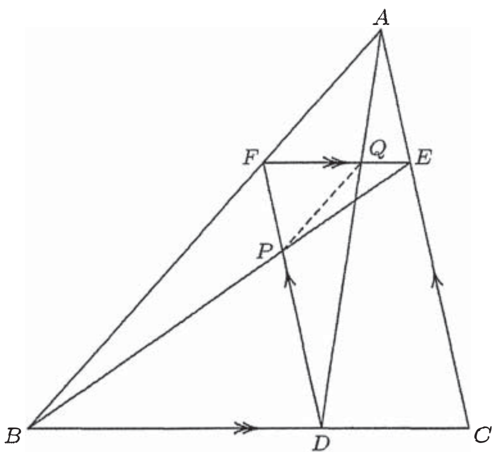
$$a+b=32, \quad a+c=36, \quad c+e=48, \quad d+e=51$$

Thus $c-b=4$, $d-c=3$ and $d-b=7$. Therefore $(a+b)+(d-b)=a+d=39$. Hence $b+c=37$. We thus have $a=15.5$, $b=16.5$, $c=20.5$, $d=23.5$ and $e=27.5$.

2. Since FE is parallel to BC and DF is parallel to CA, we have the triangles PFE, PDB and ECB are similar. Also the triangles AFQ and ABD are similar, FBD and ABC are similar. It follows that

$$\frac{DP}{PF}=\frac{BP}{PE}=\frac{BD}{DC}=\frac{BF}{FA}=\frac{DQ}{QA}$$

so that PQ is parallel to AB.



3. Let p be such a prime, then $p>2$ and is therefore odd. Thus $p=q-2=r+2$ where q,r are primes. If $r\equiv 1 \pmod 3$, then $p\equiv 0 \pmod 3$ and therefore $p=3$ and $r=1$ which is impossible. If $r\equiv 2 \pmod 3$, then $q\equiv 0 \pmod 3$ and thus $q=3$ and so $p=1$, again impossible. Thus $r\equiv 0 \pmod 3$, which means $r=3$ and hence $p=5$ and $q=7$. Thus $p=5$ is the only such prime.

4. We have $a = bm + r$ where $m = \lfloor a/b \rfloor$ and $0 \le r < b$. Thus

$$\frac{2^a+1}{2^b-1} = \frac{2^a-2^r}{2^b-1} + \frac{2^r+1}{2^b-1}.$$

Note that $2^a - 2^r = 2^r(2^{a-r}-1) = 2^r(2^{bm}-1)$, and

$$2^{bm}-1 = (2^b)^m - 1 = (2^b-1)[(2^b)^{m-1} + (2^b)^{m-2} + \cdots + 1].$$

Therefore $\frac{2^a-2^r}{2^b-1}$ is an integer.

Observe that if $b > 2$, then $2^{b-1}(2-1) > 2$, i.e.,

$$2^r + 1 \le 2^{b-1} + 1 < 2^b - 1.$$

Therefore $\frac{2^r+1}{2^b-1}$ is not an integer. Thus $\frac{2^a+1}{2^b-1}$ is not an integer.

5. Let the musicians be A, B, C, D, E, F. We first show that four concerts are sufficient. The four concerts with the performing musicians: $\{A, B, C\}$, $\{A, D, E\}$, $\{B, D, F\}$ and $\{C, E, F\}$ satisfy the requirement. We shall now prove that 3 concerts are not sufficient. Suppose there are only three concerts. Since everyone must perform at least once, there is a concert where two of the musicians, say A, B, played. But they must also play for each other. Thus we have A played and B listened in the second concert and vice versa in the third. Now C, D, E, F must all perform in the second and third concerts since these are the only times when A and B are in the audience. It is not possible for them to perform for each other in the first concert. Thus the minimum is 4.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Senior Section)

Tuesday, 4 June 2013 9:30 — 12:00

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer on the answer sheet and shade the appropriate bubbles below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. A shop sells two kinds of products A and B. One day, a salesman sold both A and B at the same price \$2100 to a customer. Suppose product A makes a profit of 20% but product B makes a loss of 20%. Then this deal

   (A) make a profit of \$70; (B) make a loss of \$70; (C) make a profit of \$175;
   (D) make a loss of \$175; (E) makes no profit or loss.

2. How many integer solutions does the equation $(x^3 - x - 1)^{x+2013} = 1$ have?

   (A) 0; (B) 1; (C) 2; (D) 3; (E) More than 3.

3. In the xy-plane, which of the following is the reflection of the graph of

$$y = \frac{1+x}{1+x^2}$$

   about the line $y = 2x$?

   (A) $x = \frac{1+y}{1+y^2}$; (B) $x = \frac{-1+y}{1+y^2}$; (C) $x = -\frac{1+y}{1+y^2}$; (D) $x = \frac{1-y}{1+y^2}$;
   (E) None of the above.

4. Let n be a positive integer. Find the number of possible remainders when

$$2013^n - 1803^n - 1781^n + 1774^n$$

   is divided by 203.

   (A) 1; (B) 2; (C) 3; (D) 4; (E) More than 4.

5. Find the number of integers n such that the equation

$$xy^2 + y^2 - x - y = n$$

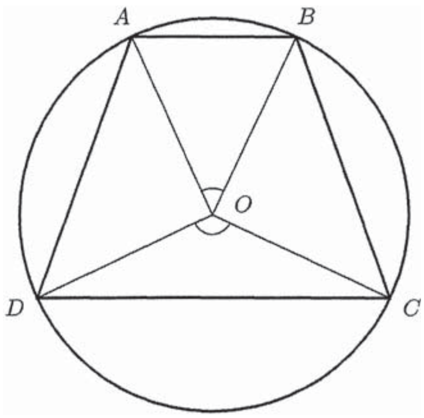
   has an infinite number of integer solutions (x, y).

   (A) 0; (B) 1; (C) 2; (D) 3; (E) More than 3.

6. If $0 < \theta < \frac{\pi}{4}$ is such that $\operatorname{cosec}\theta - \sec\theta = \frac{\sqrt{13}}{6}$, then $\cot\theta - \tan\theta$ equals

   (A) $\frac{\sqrt{13}}{6}$; (B) $\frac{\sqrt{12}}{6}$; (C) $\frac{\sqrt{5}}{6}$; (D) $\frac{13}{6}$; (E) $\frac{5}{6}$.

7. $ABCD$ is a trapezium inscribed in a circle centred at O. It is given that $AB \parallel CD$, $\angle COD = 3\angle AOB$, and $\dfrac{AB}{CD} = \dfrac{2}{5}$.

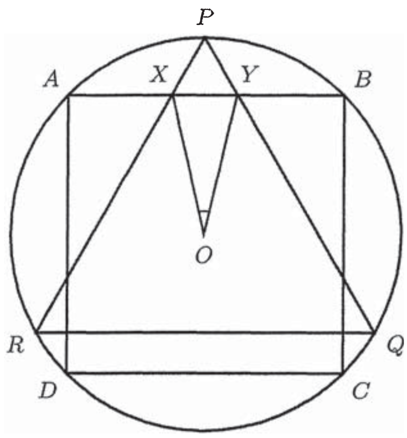


Find the ratio

$$\frac{\text{area of } \triangle BOC}{\text{area of } \triangle AOB}.$$

(A) $\dfrac{3}{2}$; (B) $\dfrac{7}{4}$; (C) $\dfrac{\sqrt{3}}{\sqrt{2}}$; (D) $\dfrac{\sqrt{5}}{2}$; (E) $\dfrac{\sqrt{7}}{\sqrt{2}}$.

8. A square $ABCD$ and an equilateral triangle PQR are inscribed in a circle centred at O in such a way that $AB \parallel QR$. The sides PQ and PR of the triangle meet the side AB of the square at X and Y respectively.



The value of $\tan \angle XOY$ is

(A) $\dfrac{1}{\sqrt{3}}$; (B) 1; (C) $\dfrac{\sqrt{6}-\sqrt{3}}{\sqrt{2}}$; (D) $\dfrac{2\sqrt{2}-2}{\sqrt{3}}$; (E) $\sqrt{3}$.

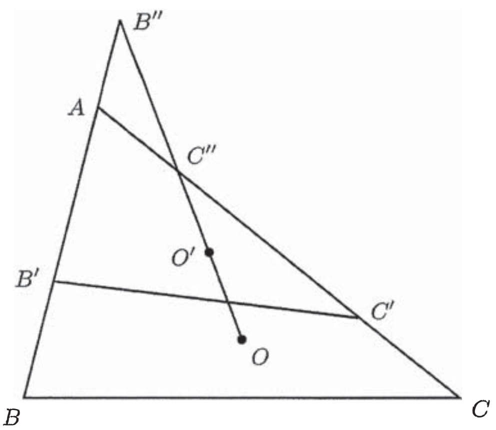
9. Two people go to the same swimming pool between 2:00p.m. and 5:00p.m. at random time and each swims for one hour. What is the chance that they meet?

(A) $\frac{1}{9}$; (B) $\frac{2}{9}$; (C) $\frac{1}{3}$; (D) $\frac{4}{9}$; (E) $\frac{5}{9}$.

10. Given a triangle $\triangle ABC$, let B' and C' be points on the sides AB and AC such that $BB' = CC'$. Let O and O' be the circumcentres (i.e., the centre of the circumscribed circle) of $\triangle ABC$ and $\triangle AB'C'$, respectively. Suppose OO' intersect lines AB' and AC' at B'' and C'', respectively. If $AB = \frac{1}{2}AC$, then

(A) $AB'' < \frac{1}{2}AC''$; (B) $AB'' = \frac{1}{2}AC''$; (C) $\frac{1}{2}AC'' < AB'' < AC''$;

(D) $AB'' = AC''$; (E) $AB'' > AC''$.



Short Questions

11. Suppose a right-angled triangle is inscribed in a circle of radius 100. Let α and β be its acute angles. If $\tan\alpha = 4\tan\beta$, find the area of the triangle.

12. Let $f(x) = \dfrac{1+10x}{10-100x}$. Set $f^n = \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ terms}}$. Find the value of

$$f\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + f^3\left(\frac{1}{2}\right) + \cdots + f^{6000}\left(\frac{1}{2}\right).$$

13. Let AB and CD be perpendicular segments intersecting at point P. Suppose that $AP = 2$, $BP = 3$ and $CP = 1$. If all the points A, B, C, D lie on a circle, find the length of DP.

14. On the xy-plane, let S denote the region consisting of all points (x, y) for which

$$\left|x+\frac{1}{2}y\right| \leq 10, \quad |x| \leq 10 \quad \text{and} \quad |y| \leq 10.$$

The largest circle centred at $(0,0)$ that can be fitted in the region S has area $k\pi$. Find the value of k.

15. Given that $\sqrt[3]{17-\frac{27}{4}\sqrt{6}}$ and $\sqrt[3]{17+\frac{27}{4}\sqrt{6}}$ are the roots of the equation

$$x^2 - ax + b = 0,$$

find the value of ab.

16. Find the number of integers between 1 and 2013 with the property that the sum of its digits equals 9.

17. Let $p(x)$ be a polynomial with integer coefficients such that $p(m) - p(n)$ divides $m^2 - n^2$ for all integers m and n. If $p(0) = 1$ and $p(1) = 2$, find the largest possible value of $p(100)$.

18. Find the number of positive integer pairs (a, b) satisfying $a^2 + b^2 < 2013$ and $a^2 b \mid (b^3 - a^3)$.

19. Let f and g be functions such that for all real numbers x and y,

$$g(f(x+y)) = f(x) + (x+y)g(y).$$

Find the value of $g(0) + g(1) + \cdots + g(2013)$.

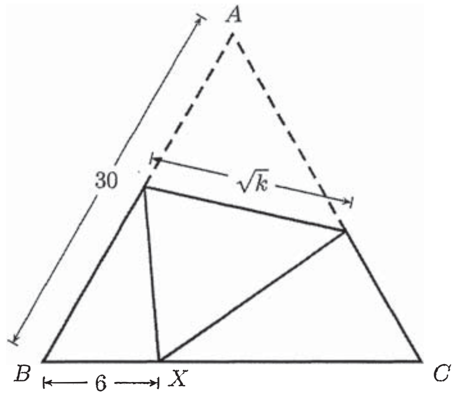
20. Each chocolate costs 1 dollar, each licorice stick costs 50 cents and each lolly costs 40 cents. How many different combinations of these three items cost a total of 10 dollars?

21. Let $A = \{1, 2, 3, 4, 5, 6\}$. Find the number of distinct functions $f : A \to A$ such that $f(f(f(n))) = n$ for all $n \in A$.

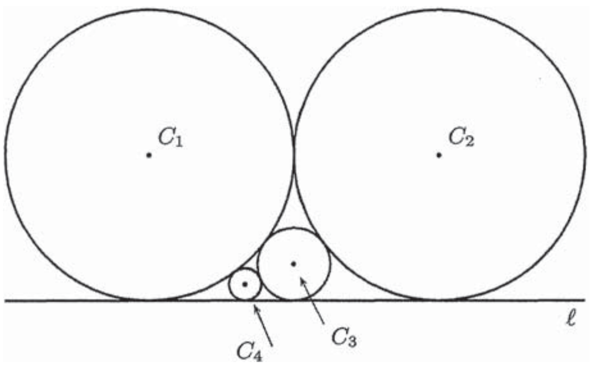
22. Find the number of triangles whose sides are formed by the sides and the diagonals of a regular heptagon (7-sided polygon). (Note: The vertices of triangles need not be the vertices of the heptagon.)

23. Six seats are arranged in a circular table. Each seat is to be painted in red, blue or green such that any two adjacent seats have different colours. How many ways are there to paint the seats?

24. $\triangle ABC$ is an equilateral triangle of side length 30. Fold the triangle so that A touches a point X on BC. If $BX = 6$, find the value of k, where $\sqrt{k}$ is the length of the crease obtained from folding.



25. As shown in the figure below, circles C_1 and C_2 of radius 360 are tangent to each other, and both tangent to straight line ℓ. If circle C_3 is tangent to C_1, C_2 and ℓ, and circle C_4 is tangent to C_1, C_3 and ℓ, find the radius of C_4.



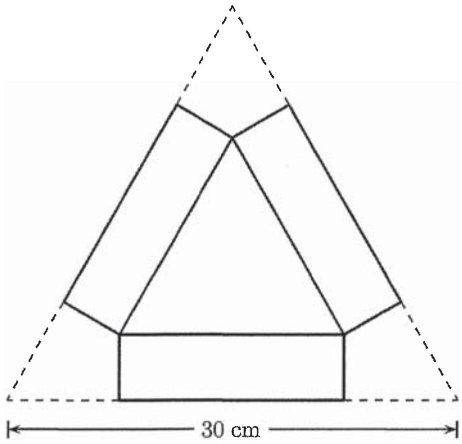
26. Set $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Find the number of real solutions to the equation

$$\{x\} + \{x^2\} = 1, \quad |x| \leq 10.$$

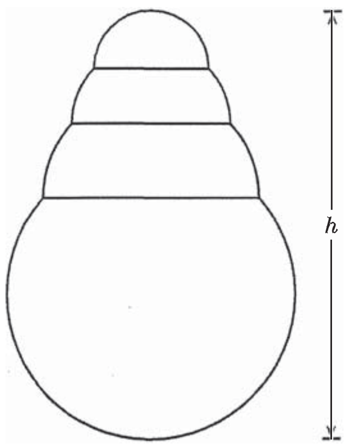
27. Find the value of $\left\lfloor \left(\dfrac{3+\sqrt{17}}{2} \right)^6 \right\rfloor$.

28. A regular dodecagon (12-sided polygon) is inscribed in a circle of radius 10. Find its area.

29. A triangular box is to be cut from an equilateral triangle of length 30 cm. Find the largest possible volume of the box (in cm^3).



30. A hemisphere is placed on a sphere of radius 100 cm. The second hemisphere is placed on the first one, and the third hemisphere is placed on the second one (as shown below). Find the maximum height of the tower (in cm).

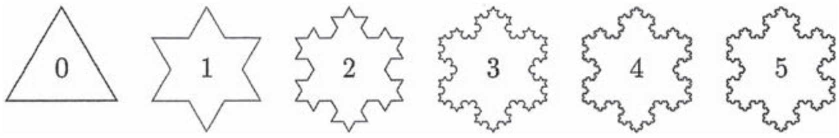


31. Let x, y, z be real numbers such that

$$x + y + z = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 1.$$

Let m denote the minimum value of $x^3 + y^3 + z^3$. Find $9m$.

32. Given an equilateral triangle of side 10, divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part. Repeat this step for each side of the resulting polygon. Find S^2, where S is the area of region obtained by repeating this procedure infinitely many times.



33. Suppose

$$\frac{1}{2013^{1000}} = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!},$$

where n is a positive integer, $a_1, \ldots, a_n$ are nonnegative integers such that $a_k < k$ for $k = 2, \ldots, n$ and $a_n > 0$. Find the value of n.

34. Let M be a positive integer. It is known that whenever $|ax^2 + bx + c| \leq 1$ for all $|x| \leq 1$, then $|2ax + b| \leq M$ for all $|x| \leq 1$. Find the smallest possible value of M.

35. Consider integers $\{1, 2, \ldots, 10\}$. A particle is initially at 1. It moves to an adjacent integer in the next step. What is the expected number of steps it will take to reach 10 for the first time?

Solutions

1. Answer: (D).

   The original values are $2100 \div 120\% = 1750$ and $2100 \div 80\% = 2625$ respectively. Then the profit is

   $$1750 + 2625 - 2 \times 2100 = -175.$$

2. Answer: (D).

   If $x + 2013 = 0$, then $x = -2013$. Suppose $x + 2013 \neq 0$. Then $x^3 - x - 1 = \pm 1$.

   If $x^3 - x - 1 = 1$, there is no integer solution; if $x^3 - x - 1 = -1$, then $x = 0, 1, -1$. Since $x + 2013$ is even, $x = 1$ or $x = -1$.

3. Answer: (E).

   (A) and (B) are the reflections with respect to $y = x$ and $y = -x$ respectively; (C) and (D) are the rotations about the origin by $90°$ and $-90°$ respectively.

4. Answer: (A).

   For any positive integer n,

   $$\begin{aligned} 2013^n - 1803^n - 1781^n + 1774^n &= (2013^n - 1803^n) - (1781^n - 1774^n) \\ &= (2013 - 1803)u - (1781 - 1774)v = 210u - 7v, \\ 2013^n - 1803^n - 1781^n + 1774^n &= (2013^n - 1781^n) - (1803^n - 1774^n) \\ &= (2013 - 1781)x - (1803 - 1774)y = 29x - 29y. \end{aligned}$$

   So $2013^n - 1803^n - 1781^n + 1774^n$ is divisible by $7 \times 29 = 203$ for every positive integer n.

5. Answer: (C).

   Rewrite the equation as

   $$xy^2 + y^2 - x - y = (y - 1)(x(y + 1) + y) = n.$$

   If $n = 0$, then there are infinitely many integer solutions. Suppose $n \neq 0$, and the equation has infinitely many integer solutions. Then there exists a divisor k of n such that $y - 1 = k$ and $x(y + 1) + y = n/k$ for infinitely many x. It forces $y + 1 = 0$, i.e., $y = -1$. Then $n = 2$.

6. Answer: (E).

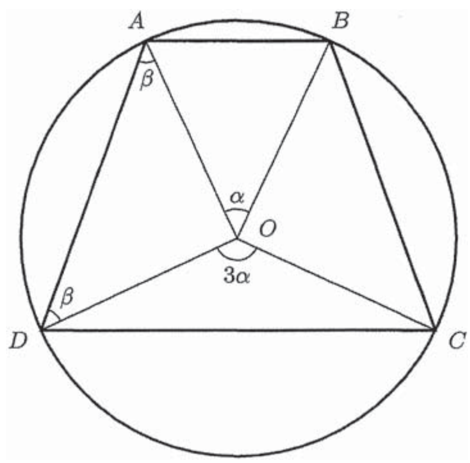
Let $k = \sin\theta\cos\theta$. Then

$$\frac{13}{36} = \left(\frac{1}{\sin\theta} - \frac{1}{\cos\theta}\right)^2 = \frac{1 - 2\sin\theta\cos\theta}{(\sin\theta\cos\theta)^2} = \frac{1-2k}{k^2}.$$

Solve the equation: $k = 6/13$ ($k = -6$ is rejected). Then $\sin 2\theta = 2k = 12/13$ and

$$\cot\theta - \tan\theta = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta} = \frac{5/13}{6/13} = \frac{5}{6}.$$

7. Answer: (A).



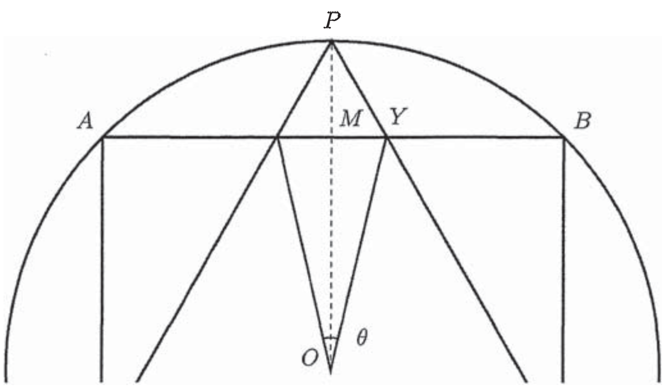
Let $\angle AOB = \alpha$ and $\angle ADO = \beta$. Then $\frac{1}{2}(\alpha + 3\alpha) = 2\beta$; that is, $\alpha = \beta$. Given that

$$\frac{5}{2} = \frac{\sin(3\alpha/2)}{\sin(\alpha/2)} = \frac{3\sin(\alpha/2) - 4\sin^3(\alpha/2)}{\sin(\alpha/2)} = 3 - 4\sin^2(\alpha/2).$$

Then $\sin^2(\alpha/2) = \frac{1}{8}$. Hence,

$$\frac{S_{\triangle BOC}}{S_{\triangle AOB}} = \frac{\sin(\pi - 2\alpha)}{\sin\alpha} = 2\cos\alpha = 2\left(1 - 2\sin^2(\alpha/2)\right) = \frac{3}{2}.$$

8. Answer: (C).



Let the side of the square be 2. Then the radius of the circle is $\sqrt{2}$. Let $\theta = \angle XOY$. So

$$\tan(\theta/2) = \frac{MY}{MO} = MY = PM \tan 30^\circ = \frac{\sqrt{2}-1}{\sqrt{3}}.$$

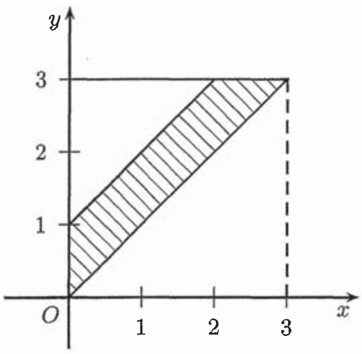
Then

$$\tan\theta = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} = \frac{\sqrt{6}-\sqrt{3}}{\sqrt{2}}.$$

9. Answer: (E).

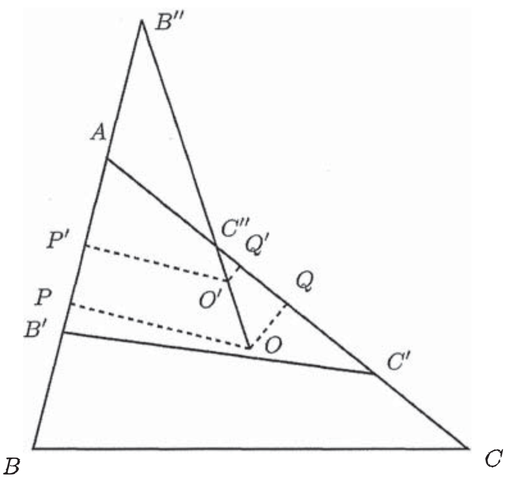
Let x and y denote the numbers of hours after 2:00p.m. that the first and the second person visits the swimming pool, respectively. Then

$$0 \le x \le y \le 3 \quad \text{and} \quad y \le x+1.$$



So the chance that they meet is $\dfrac{5/2}{9/2} = \dfrac{5}{9}$.

10. Answer: (D).



Let P and P' be the projections of O and O' on AB respectively. Then

$$PP' = AP - AP' = \frac{1}{2}AB - \frac{1}{2}AB' = \frac{1}{2}(AB - AB') = \frac{1}{2}BB'.$$

Similarly, let Q and Q' be the projections of O and O' on AC respectively, then $QQ' = \frac{1}{2}CC'$.

$$\sin\angle O'OP = \frac{PP'}{OO'} = \frac{QQ'}{OO'} = \sin\angle O'OQ \Rightarrow \angle O'OP = \angle O'OQ.$$

So $\angle AB''C'' = \angle AC''B''$. It follows that $AB'' = AC''$.

11. Answer: 8000.

$\tan\alpha = 4\tan\beta = \frac{4}{\tan\alpha} \Rightarrow \tan\alpha = 2$. The two legs are $\frac{200}{\sqrt{5}}$ and $\frac{400}{\sqrt{5}}$ respectively.

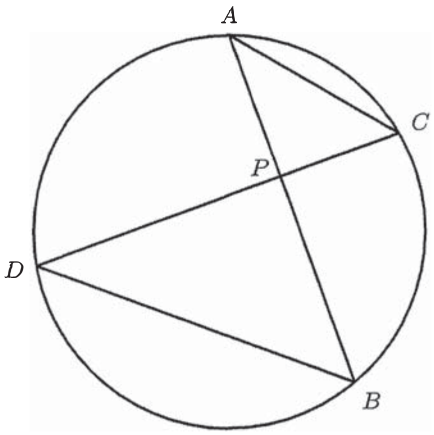
$$\text{Area} = \frac{1}{2} \times \frac{200}{\sqrt{5}} \times \frac{400}{\sqrt{5}} = 8000.$$

12. Answer: 595.

Let $f(x) = \frac{1+10x}{10-100x}$. Then $f^2(x) = -\frac{1}{100x}$, $f^3(x) = \frac{1-10x}{10+100x}$ and $f^4(x) = x$. Then

$$\begin{aligned} f\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + \cdots + f^{6000}\left(\frac{1}{2}\right) &= 1500\left[f\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + f^4\left(\frac{1}{2}\right)\right] \\ &= 1500\left(-\frac{3}{10} - \frac{1}{50} + \frac{1}{15} + \frac{1}{2}\right) = 595. \end{aligned}$$

13. Answer: 6.



Since $\angle A = \angle D$ and $\angle C = \angle B$, the triangles $\triangle ACP$ and $\triangle DBP$ are similar. Then

$$\frac{DP}{BP} = \frac{AP}{CP} \Rightarrow DP = \frac{AP}{CP} \times BP = \frac{2}{1} \times 3 = 6.$$

14. Answer: 80.

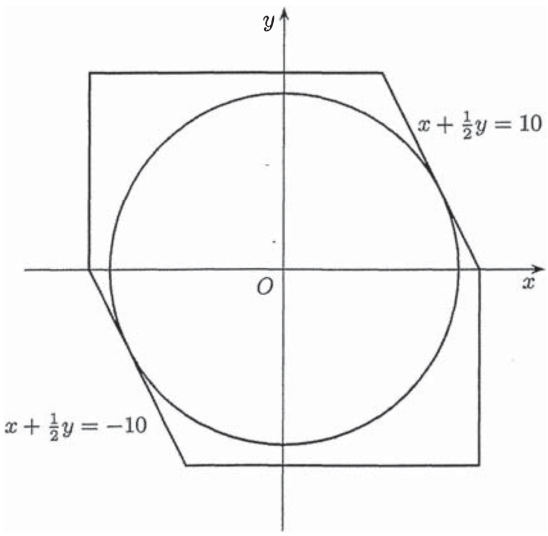
The region S is the hexagon enclosed by the lines

$$x = \pm 10, \quad y = \pm 10, \quad x + \frac{1}{2}y = \pm 10.$$

The largest circle contained in S is tangent to $x+\frac{1}{2}y=\pm 10$. Hence, its radius is the distance from the origin $(0,0)$ to $x+\frac{1}{2}y=10$:

$$r=\frac{10}{\sqrt{1+(1/2)^2}}=4\sqrt{5}.$$

The area of the largest circle is thus $\pi r^2=\pi(4\sqrt{5})^2=80\pi$.



15. Answer: 10.

Let $x_1=\sqrt[3]{17-\frac{27}{4}\sqrt{6}}$ and $x_2=\sqrt[3]{17+\frac{27}{4}\sqrt{6}}$. Then

$$b=x_1x_2=\sqrt[3]{\left(17-\frac{27}{4}\sqrt{6}\right)\left(17+\frac{27}{4}\sqrt{6}\right)}=\sqrt[3]{\frac{125}{8}}=\frac{5}{2},$$

$$\begin{aligned}a^3&=(x_1+x_2)^3=x_1^3+x_2^3+3x_1x_2(x_1+x_2)\\&=\left(17-\frac{27}{4}\sqrt{6}\right)+\left(17+\frac{27}{4}\sqrt{6}\right)+3ab=34+\frac{15}{2}a.\end{aligned}$$

Then $a=4$ and thus $ab=10$.

16. Answer: 101.

Case 1: $n<1000$. Write $n=\overline{abc}$. Then

$$a+b+c=9,\qquad a,b,c\in\{0,1,\dots,9\}.$$

Case 2: $1000\le n<2000$. Write $n=\overline{1abc}$. Then

$$a+b+c=8,\qquad a,b,c\in\{0,1,\dots,8\}.$$

Case 3: $2000\le n\le 2013$. Then $n=2007$.

Therefore, there are $\binom{9+3-1}{9}+\binom{8+3-1}{8}+1=55+45+1=101$ such numbers.

17. Answer: 10001.

Let $n = 0$. Since $p(m) - 1 \mid m^2$ for all m, $\deg p(x) \leq 2$. Let $p(x) = ax^2 + bx + c$. Then

$$\frac{p(m) - p(n)}{m - n} = \frac{a(m^2 - n^2) + b(m - n)}{m - n} = a(m + n) + b$$

divides $m + n$. If $a \neq 0$, then $a = \pm 1$ and $b = 0$; if $a = 0$, then $b = \pm 1$. Thus

$$p(x) = \pm x^2 + c, \ \pm x + c, \ c.$$

Since $p(0) = 1$ and $p(1) = 2$, we have $p(x) = x^2 + 1$ or $p(x) = x + 1$. The largest possible value of $p(100)$ is $100^2 + 1 = 10001$.

18. Answer: 31.

Let $k = \dfrac{b^3 - a^3}{a^2 b} = \left(\dfrac{b}{a}\right)^2 - \dfrac{a}{b}$. Then

$$\left(\frac{a}{b}\right)^3 + k\left(\frac{a}{b}\right)^2 - 1 = 0.$$

The only possible positive rational solution of $x^3 + kx^2 - 1 = 0$ is $x = 1$; namely, $a = b$. Conversely, if $a = b$, then it is obvious that $a^2 b \mid (b^3 - a^3)$.

Then $2013 > a^2 + b^2 = 2a^2$ implies $a \leq 31$.

19. Answer: 0.

Let $y = -x$. Then $g(f(0)) = f(x)$ for all x. This shows that f is a constant function; namely $f(x) = c$ for some c. So that $g(c) = g(f(0)) = f(x) = c$. For all x, y, we have

$$(x + y)g(y) = g(f(x + y)) - f(x) = g(c) - c = 0.$$

Since $x + y$ is arbitrary, we must have $g(y) = 0$ for all y. Hence,

$$g(0) + g(1) + \cdots + g(2013) = 0.$$

20. Answer: 36.

Let x, y and z denote the numbers of chocolate, licorice stick and lolly, respectively. Then

$$x + 0.5y + 0.4z = 10.$$

For each $k = 0, \ldots, 10$, consider $0.5y + 0.4z = k$, i.e., $5y + 4z = 10k$. Then

$$2 \mid y \quad \text{and} \quad 5 \mid z.$$

Set $y = 2s$ and $z = 5t$. Then $10s + 20t = 10k$, i.e., $s + 2t = k$. Then $t = 0, \ldots, \lfloor k/2 \rfloor$. So there are $\lfloor k/2 \rfloor + 1$ ways to use k dollars. The total number of ways is

$$\sum_{k=0}^{10} (\lfloor k/2 \rfloor + 1) = 1 + 1 + 2 + 2 + \cdots + 5 + 5 + 6 = 36.$$

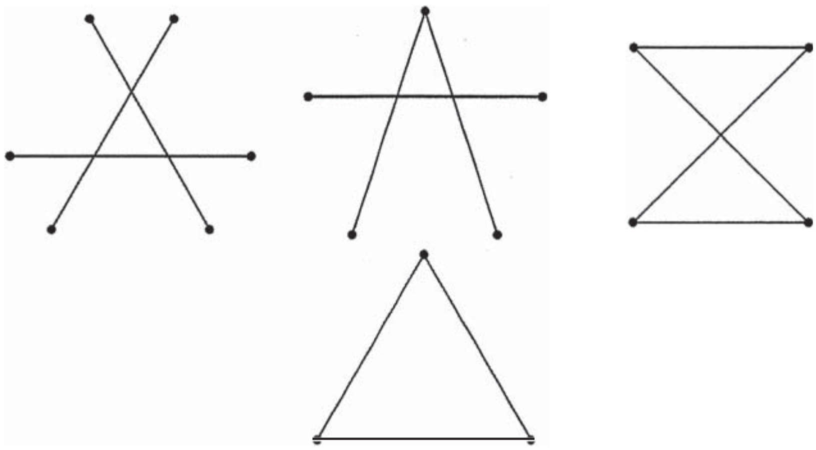
21. Answer: 81.

Suppose $f(f(f(n))) = n$ for all n. For some $k \in \{0, 1, 2\}$, there exist distinct a_i, b_i, c_i, $i \le k$, such that $f(a_i) = b_i$, $f(b_i) = c_i$ and $f(c_i) = a_i$ and $f(n) = n$ if $n \neq a_i, b_i, c_i$. So the total number of required functions is

$$\binom{6}{6} + \binom{6}{3} \times 2 + \frac{1}{2} \times \binom{6}{3}\binom{3}{3} \times 2^2 = 81.$$

22. Answer: 287.

A triangle can be formed using 3, 4, 5 or 6 vertices.



So the total number is

$$\binom{7}{6} + 5 \times \binom{7}{5} + 2 \times 2 \times \binom{7}{4} + \binom{7}{3} = 287.$$

23. Answer: 66.

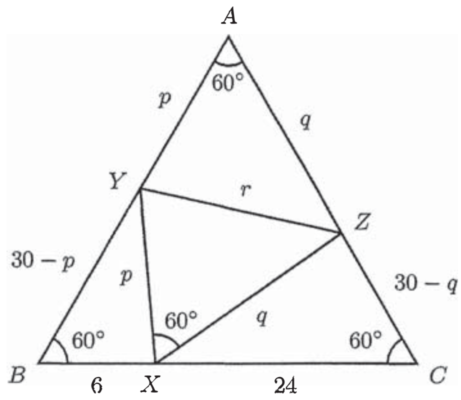
Let $n \ge 2$ be an integer, and let S_n denote the number of ways to paint n seats $a_1, \ldots, a_n$ as described, but with a_1 painted red. Consider S_{n+2} where $n \ge 2$.

Case 1: a_3 is painted red. Then there are 2 choices for a_2. Thus, the total number of ways for this case is $2S_n$.

Case 2: a_3 is not painted red. Since the colour of a_2 is uniquely determined by the colour of a_3, this is equivalent to the case when there are $(n+1)$ seats. The total number of ways for this case is S_{n+1}.

We conclude that $S_{n+2} = S_{n+1} + 2S_n$. It is clear that $S_2 = S_3 = 2$. Then $S_4 = 6$, $S_5 = 10$ and $S_6 = 22$. So the required number of ways is $3 \times 22 = 66$.

24. Answer: 343.



Apply the law of cosine on $\triangle XBY$ and $\triangle XCZ$ respectively:

$$p^2 = 6^2 + (30-p)^2 - 6(30-p),$$
$$q^2 = 24^2 + (30-q)^2 - 24(30-q).$$

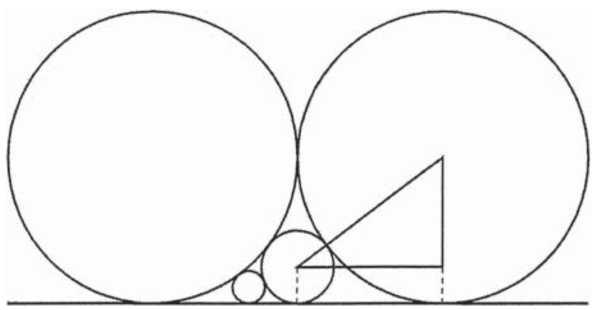
Then $p = 14$ and $q = 21$. Applying the law of cosine in $\triangle YXZ$ again to obtain

$$k = r^2 = p^2 + q^2 - pq = 14^2 + 21^2 - 14 \cdot 21 = 343.$$

25. Answer: 40.

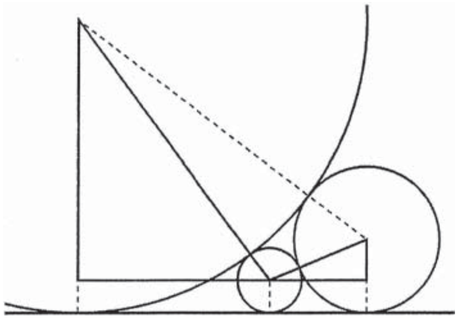
Let R be the radius of C_3. Then

$$(360 - R)^2 + 360^2 = (360 + R)^2 \Rightarrow R = 90.$$



Let r be the radius of C_4. Then

$$\sqrt{(360+r)^2 - (360-r)^2} + \sqrt{(90+r)^2 - (90-r)^2} = 360 \Rightarrow r = 40.$$



26. Answer: 181.

Since $\{x\} + \{x^2\} = 1$, $x + x^2 = n$ for some integer n. Then

$$x = \frac{-1 \pm \sqrt{1+4n}}{2}.$$

$\dfrac{-1+\sqrt{1+4n}}{2} \leq 10$ gives $0 \leq n \leq 110$; $\dfrac{-1-\sqrt{1+4n}}{2} \geq -10$ implies $0 \leq n \leq 90$.

If $\{x\} + \{x^2\} \neq 1$, then $\{x\} + \{x^2\} = 0$, which happens only if x is an integer between -10 to 10. So the total number of solutions to $\{x\} + \{x^2\} = 1$ is $111 + 91 - 21 = 181$.

27. Answer: 2040.

Let $\alpha = \dfrac{3+\sqrt{17}}{2}$ and $\beta = \dfrac{3-\sqrt{17}}{2}$. Then $\alpha\beta = -2$ and $\alpha + \beta = 3$.

Set $S_n = \alpha^n + \beta^n$. Then

$$\begin{aligned} 3S_{n+1} + 2S_n &= (\alpha+\beta)(\alpha^{n+1} + \beta^{n+2}) - \alpha\beta(\alpha^n + \beta^n) \\ &= \alpha^{n+2} - \beta^{n+2} = S_{n+2}. \end{aligned}$$

Note that $|\beta| < 1$. Then for even positive integer n, $\lfloor \alpha^n \rfloor = S_n + \lfloor -\beta^n \rfloor = S_n - 1$.

Since $S_0 = 2$ and $S_1 = 3$, we can proceed to evaluate that $S_6 = 2041$.

28. Answer: 300.

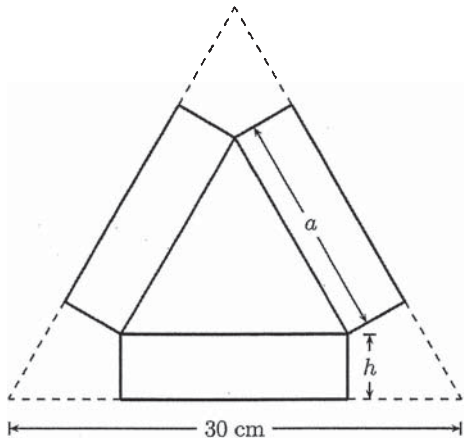
The area is $12 \times \dfrac{1}{2} \times 10^2 \times \sin 30^\circ = 300$.

29. Answer: 500.

Let the length and the height of the box be a and h, respectively. Note that $a + 2\sqrt{3}\,h = 30$. Then the volume of the box is

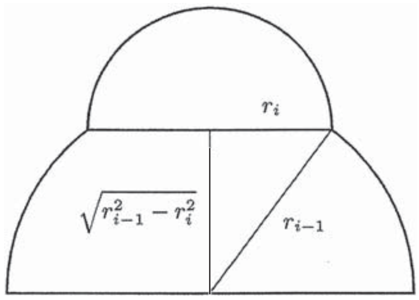
$$\frac{\sqrt{3}\,a^2}{4} h = \frac{1}{2}\left(\frac{a}{2} \cdot \frac{a}{2} \cdot 2\sqrt{3}\,h\right) \leq \frac{1}{2}\left(\frac{a + 2\sqrt{3}h}{3}\right)^3 = \frac{1}{2} \times 10^3 = 500.$$

The equality holds if $a/2 = 2\sqrt{3}h$, i.e., $a = 20$ and $h = 5/\sqrt{3}$.



30. Answer: 300.

Let the radius of the ith hemisphere be r_i metre ($r_0 = 1$). Set $h_i = \sqrt{r_{i-1}^2 - r_i^2}$.



By Cauchy inequality,

$$\left(\sqrt{r_0^2 - r_1^2} + \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + r_3\right)^2 \leq 4(r_0^2 - r_1^2 + r_1^2 - r_2^2 + r_2^2 - r_3^2 + r_3^2)$$
$$= 4r_0^2 = 4.$$

The total height $h \leq r_0 + \sqrt{r_0^2 - r_1^2} + \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + r_3 \leq 1 + \sqrt{4} = 3\,\text{m} = 300\,\text{cm}$.

The equality holds if $r_i = \dfrac{\sqrt{4-i}}{2}$, $i = 0, 1, 2, 3$.

31. Answer: 5.

It is clear that $|x|, |y|, |z| \leq 1$. Note that

$$0 = x + y + z - x^2 - y^2 - z^2 = x(1-x) + y(1-y) + z(1-z).$$

Without loss of generality, assume that $z \leq 0$. Then $x = (1-y) + (-z) \geq 0$ and similarly $y \geq 0$. Since

$$1 - z^2 = x^2 + y^2 \geq \frac{(x+y)^2}{2} = \frac{(1-z)^2}{2},$$

we have $-1/3 \le z \le 0$. On the other hand,

$$\begin{aligned} x^3+y^3+z^3 &= (x+y)\left[\frac{3(x^2+y^2)-(x+y)^2}{2}\right]+z^3 \\ &= (1-z)\left[\frac{3(1-z^2)-(1-z)^2}{2}\right]+z^3 = 1-3z^2+3z^3 \end{aligned}$$

increases as z increases. The minimum $m = 5/9$ is obtained at $z = -1/3$ and $x = y = 2/3$.

32. Answer: 4800.

Let S_n denote the area of the region obtained in the nth step. Then $S_0 = 25\sqrt{3}$ and $S_n - S_{n-1} = \dfrac{3}{4}\left(\dfrac{4}{9}\right)^n S_0$ for all $n \ge 1$. Then

$$\begin{aligned} S_n &= S_0 + \frac{3}{4}\left[\frac{4}{9}+\left(\frac{4}{9}\right)^2+\cdots+\left(\frac{4}{9}\right)^n\right]S_0 \\ &= S_0\left[1+\frac{3}{4}\cdot\frac{4}{9}\cdot\frac{1-(4/9)^n}{1-4/9}\right] = 25\sqrt{3}\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right]. \end{aligned}$$

As n increases, S_n tends to $S = 25\sqrt{3}\cdot\dfrac{8}{5} = 40\sqrt{3}$. So $S^2 = 4800$.

33. Answer: 60024.

Let $x = \dfrac{1}{2013^{1000}}$. Then

$$\begin{aligned} xn! &= a_1 n! + a_2\frac{n!}{2!}+\cdots+a_{n-1}\frac{n!}{(n-1)!}+a_n\frac{n!}{n!}, \\ x(n-1)! &= a_1(n-1)!+a_2\frac{(n-1)!}{2!}+\cdots+a_{n-1}\frac{(n-1)!}{(n-1)!}+\frac{a_n}{n}. \end{aligned}$$

So n is the smallest integer such that $n!x$ is an integer, i.e., $2013^{1000} \mid n!$, or equivalently $61^{1000} \mid n!$ because 61 is the largest prime divisor of 2013.

Since $\left\lfloor \dfrac{1000}{61} \right\rfloor = 16$, $n = (1000-16)\times 61 = 60024$.

34. Answer: 4.

Let a, b, c be fixed. Set $f(x) = ax^2+bx+c$. Then

$$f(-1) = a-b+c, \quad f(0) = c, \quad f(1) = a+b+c.$$

Solve the system to get

$$a = \frac{1}{2}f(-1)-f(0)+\frac{1}{2}f(1), \qquad b = -\frac{1}{2}f(-1)+\frac{1}{2}f(1).$$

Suppose $|f(x)| \leq 1$ for all $|x| \leq 1$. Then

$$\begin{aligned}|2ax+b| &= \left|\left(x-\frac{1}{2}\right)f(-1) - 2f(0)x + \left(x+\frac{1}{2}\right)f(1)\right| \\ &\leq \left|x-\frac{1}{2}\right| + 2|x| + \left|x+\frac{1}{2}\right| \\ &\leq \left|x-\frac{1}{2}\right| + \left|x+\frac{1}{2}\right| + 2 \leq 4.\end{aligned}$$

Moreover, $|2x^2-1| \leq 1$ whenever $|x| \leq 1$, and $|2x| = 4$ is achieved at $x = \pm 1$.

35. Answer: 81.

Let E_k denote the expected number of steps it takes to go from $k-1$ to k, $k = 2, \ldots, 100$. Then $E_{k+1} = \frac{1}{2}(1 + E_k + E_{k+1}) + \frac{1}{2}$, which implies $E_{k+1} = E_k + 2$.

It is clear that $E_2 = 1$. Then $E_3 = 3, E_4 = 5, \ldots, E_{10} = 17$. So

$$E = E_2 + E_3 + \cdots + E_{10} = 1 + 3 + \cdots + 17 = 81.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Senior Section, Round 2)

Saturday, 29 June 2013 **0900-1300**

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1. In the triangle ABC, $AB > AC$, the extension of the altitude AD with D lying inside BC intersects the circumcircle ω of the triangle ABC at P. The circle through P and tangent to BC at D intersects ω at Q distinct from P with $PQ = DQ$. Prove that $AD = BD - DC$.

2. Find all pairs of integers (m, n) such that

$$m^3 - n^3 = 2mn + 8.$$

3. Let $b_1, b_2, \ldots$ be a sequence of positive real numbers such that for each $n \geq 1$,

$$b_{n+1}^2 \geq \frac{b_1^2}{1^3} + \frac{b_2^2}{2^3} + \cdots + \frac{b_n^2}{n^3}.$$

Show that there is a positive integer M such that

$$\sum_{n=1}^{M} \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} > \frac{2013}{1013}.$$

4. In the following 6×6 array, one can choose any $k \times k$ subarray, with $1 < k \leq 6$ and add 1 to all its entries. Is it possible to perform the operation a finite number of times so that all the entries in the array are multiples of 3?

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

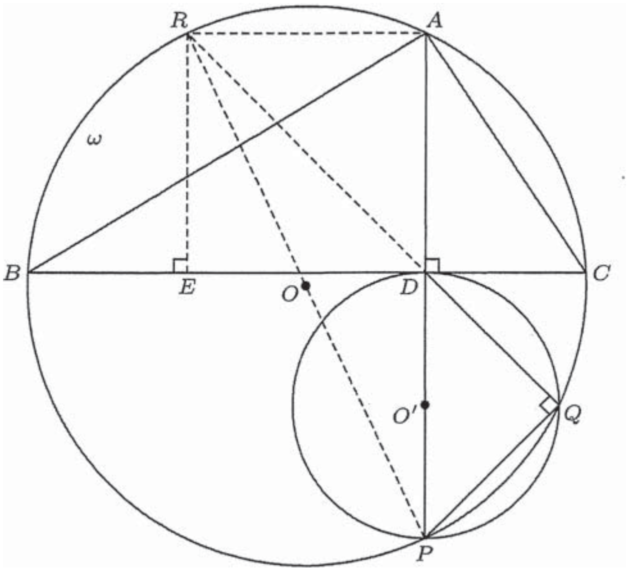
5. Let x, y be distinct real numbers such that $\frac{x^n - y^n}{x-y}$ is an integer for four consecutive positive integers n. Prove that $\frac{x^n - y^n}{x-y}$ is an integer for all positive integers n.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Senior Section, Round 2 solutions)

1.



Let the extension of QD meet ω at R. Since $\angle PQR = 90°$, PR is a diameter of ω. Thus $\angle PAR = 90°$ so that RA is parallel to BC. This means $BCAR$ is an isosceles trapezoid. Let E be the foot of the perpendicular from R onto BC. Then $BE = CD$ and $ARED$ is a rectangle. Since $\angle ADR = 45°$, $ARED$ is in fact a square so that $AD = DE$. Therefore, $BD - DC = BD - BE = DE = AD$.

2. When $m = 0$, $n = -2$ and when $n = 0$, $m = 2$. These are the two obvious solutions: $(m, n) = (0, -2), (2, 0)$. We'll show that there are no solutions when $mn \neq 0$.

Suppose $mn < 0$. If $m > 0, n < 0$, then

$$-2m|n| + 8 = m^3 + |n|^3 \geq m^2 + |n|^2 \quad \Rightarrow \quad 8 \geq (m + |n|)^2.$$

Thus $m + |n| = 2$ or $m = 1$, $n = -1$ and this is not a solution. If $m < 0$ and $n > 0$, then

$$-2|m|n + 8 = -|m|^3 - n^3 \leq -|m|^2 - |n|^2 \quad \Rightarrow \quad 8 \leq -(|m| - n)^2$$

which is impossible.

3. From Cauchy -Schwarz inequality, we have

$$
\begin{aligned}
&\left(\sum_{k=1}^{n}\frac{b_k^2}{k^3}\right)\left(\sum_{k=1}^{n}k^3\right) \geq \left(\sum_{k=1}^{n}b_k^2\right)^2 \\
\Rightarrow\quad &\left(\sum_{k=1}^{n}\frac{b_k^2}{k^3}\right)\left(\frac{n(n+1)}{2}\right)^2 \geq \left(\sum_{k=1}^{n}b_k^2\right)^2 \\
\Rightarrow\quad &\frac{b_{n+1}}{b_1+b_2+\cdots+b_n} \geq \frac{2}{n(n+1)} \\
\Rightarrow\quad &\sum_{n=1}^{M}\frac{b_{n+1}}{b_1+b_2+\cdots+b_n} \geq \sum_{n=1}^{M}\frac{2}{n(n+1)} \\
\Rightarrow\quad &\sum_{n=1}^{M}\frac{b_{n+1}}{b_1+b_2+\cdots+b_n} \geq 2\sum_{n=1}^{M}\frac{1}{n}-\frac{1}{n+1} \\
&\qquad\qquad\qquad\qquad\qquad = 2-\frac{2}{M+1} \geq \frac{2013}{1013}
\end{aligned}
$$

if $M \geq 155$.

4. The answer is no. Let the original array be A. Consider the following array

$$
M = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{bmatrix}.
$$

Multiply the corresponding elements of the two arrays and compute the sum modulo 3. It's easy to verify that this sum is invariant under the given operation. Since the original sum is 2, one can never obtain an array where all the entries are multiples of 3.

5. Note that

$$(x+y)(x^n-y^n) = (x^{n+1}-y^{n+1}) + xy(x^{n-1}-y^{n-1}).$$

Let $t_n = \frac{x^n-y^n}{x-y}$. Then $t_0 = 0$, $t_1 = 1$ and for $n \geq 0$, $t_{n+2} + bt_{n+1} + ct_n = 0$, with $b = -(x+y)$ and $c = xy$. Then it suffices to show that $b, c \in \mathbb{Z}$.

If $c = 0$, then either $x = 0$ or $y = 0$. Say $y = 0$. Then $t_n = x^{n-1}$, with $x \neq 0$. Then $x = \frac{t_{m+1}}{t_m} \in \mathbb{Q}$. From $t_{m+1} = x^m \in \mathbb{Z}$, it follows that $x \in \mathbb{Z}$. Thus $t_n \in \mathbb{Z}$ for all n. The case $x = 0$ is similar.

We now assume that $c \neq 0$. Let $t_n \in \mathbb{Z}$ for $n = m, m+1, m+2, m+3$. Note that $c^n = (xy)^n = t_{n+1}^2 - t_n t_{n+2}$. Thus $c^m, c^{m+1} \in \mathbb{Z}$. Therefore $c = \frac{c^{m+1}}{c^m} \in \mathbb{Q}$. As before, we have $c \in \mathbb{Z}$. If both t_{m+1}, t_{m+2} are 0, then using the recurrence, we can show easily that $t_n = 0$ for all n, a contradiction. Thus one of them is nonzero. Note that, with $k = m+1$ or $m+2$, whichever is nonzero, we have

$$b = \frac{-ct_{k-1} - t_{k+1}}{t_k} \in \mathbb{Q}.$$

From the recurrence, it follows by induction that t_n can be represented as $t_n = f_{n-1}(b)$ where $f_{n-1}(X)$ is a polynomial with integer coefficients, $\deg f_{n-1} = n-1$ and with the coefficient of $X^{n-1} = \pm 1$. Since $b \in \mathbb{Q}$ is a root of the equation $f_m(X) = t_{m+1}$, $b \in \mathbb{Z}$.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2013
(Open Section, First round)

Wednesday, 5 June 2013 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

1. The sum

$$\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+\frac{1}{3\times 4\times 5}+\cdots+\frac{1}{100\times 101\times 102}$$

can be expressed as $\frac{a}{b}$, a fraction in its simplest form. Find $a+b$.

2. Determine the maximum value of $\frac{1+\cos x}{\sin x+\cos x+2}$, where x ranges over all real numbers.

3. Let $\tan\alpha$ and $\tan\beta$ be two solutions of the equation $x^2-3x-3=0$. Find the value of

$$\left|\sin^2(\alpha+\beta)-3\sin(\alpha+\beta)\cos(\alpha+\beta)-3\cos^2(\alpha+\beta)\right|.$$

(Note: $|x|$ denotes the absolute value of x.)

4. Suppose that $a_1, a_2, a_3, a_4, \cdots$ is an arithmetic progression with $a_1>0$ and $3a_8=5a_{13}$. Let $S_n=a_1+a_2+\cdots+a_n$ for all integers $n\geq 1$. Find the integer n such that S_n has the maximum value.

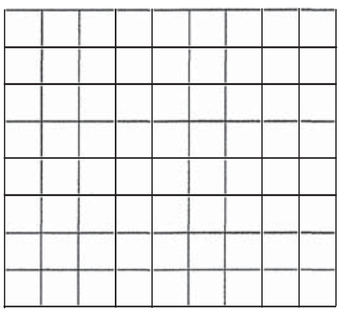
5. If $g(x)=\tan\frac{x}{2}$ for $0<x<\pi$ and $f(g(x))=\sin 2x$, find the value of k such that $kf(\frac{\sqrt{2}}{2})=36\sqrt{2}$.

6. Let $g(x)$ be a strictly increasing function defined for all $x\geq 0$. It is known that the range of t satisfying

$$g(2t^2+t+5)<g(t^2-3t+2)$$

is $b<t<a$. Find $a-b$.

7. The figure below shows an 8×9 rectangular board.



How many squares are there in the above rectangular board?

8. Let a, b, c be positive real numbers such that $a+b+c=2013$. Find the maximum value of $\sqrt{3a+12}+\sqrt{3b+12}+\sqrt{3c+12}$.

9. Let $A=\cos^2 10^\circ+\cos^2 50^\circ-\sin 40^\circ\sin 80^\circ$. Determine the value of $100A$.

10. Assume that $a_i \in \{1, -1\}$ for all $i = 1, 2, \cdots, 2013$. Find the least positive number of the following expression

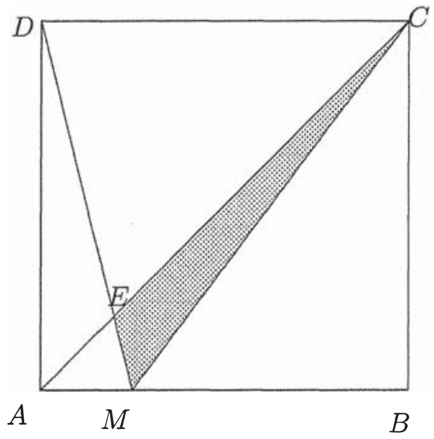
$$\sum_{1 \leq i < j \leq 2013} a_i a_j.$$

11. Let f be a function defined on non-zero real numbers such that

$$\frac{27\mathrm{f}(-x)}{x} - x^2 \mathrm{f}\left(\frac{1}{x}\right) = -2x^2,$$

for all $x \neq 0$. Find f(3).

12. In the figure below, $ABCD$ is a square with $AB = 20$ cm (not drawn to scale). Assume that M is a point such that the area of the shaded region is 40 cm^2. Find AM in centimetres.



13. In the triangle ABC, a circle passes through the point A, the midpoint E of AC, the midpoint F of AB and is tangent to the side BC at D. Suppose

$$\frac{AB}{AC} + \frac{AC}{AB} = 4.$$

Determine the size of $\angle EDF$ in degrees.

14. Let $a_1, a_2, a_3, \cdots$ be a sequence of real numbers in a geometric progression. Let $S_n = a_1 + a_2 + \cdots + a_n$ for all integers $n \geq 1$. Assume that $S_{10} = 10$ and $S_{30} = 70$. Find the value of S_{40}.

15. Find the number of three-digit numbers which are multiples of 3 and are formed by the digits 0,1,2,3,4,5,6,7 without repetition.

16. All the positive integers which are co-prime to 2012 are grouped in an increasing order in such a way that the n^{th} group has $2n - 1$ numbers. So, the first three groups in this grouping are $(1), (3, 5, 7), (9, 11, 13, 15, 17)$. It is known that 2013 belongs to the k^{th} group. Find the value of k.

    (Note: Two integers are said to be co-prime if their greatest common divisor is 1.)

17. The numbers $1, 2, 3, \cdots, 7$ are randomly divided into two non-empty subsets. The probability that the sum of the numbers in the two subsets being equal is $\frac{p}{q}$ expressed in the lowest term. Find $p + q$.

18. Find the number of real roots of the equation $\log_{10}^2 x - \lfloor \log_{10} x \rfloor - 2 = 0$.

    (Note: $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.)

19. In the triangle ABC, $AB = AC$, $\angle A = 90°$, D is the midpoint of BC, E is the midpoint of AC and F is a point on AB such that BE intersects CF at P and B, D, P, F lie on a circle. Let AD intersect CP at H. Given $AP = \sqrt{5} + 2$, find the length of PH.

20. Find the total number of positive integers n not more than 2013 such that $n^4 + 5n^2 + 9$ is divisible by 5.

21. In a circle ω centred at O, AA' and BB' are diameters perpendicular to each other such that the points A, B, A', B' are arranged in an anticlockwise sense in this order. Let P be a point on the minor arc $A'B'$ such that AP intersects BB' at D and BP intersects AA' at C. Suppose the area of the quadrilateral $ABCD$ is 100. Find the radius of ω.

22. A sequence $a_1, a_2, a_3, a_4, \cdots$, with $a_1 = \frac{1}{2}$, is defined by

$$a_n = 2a_n a_{n+1} + 3a_{n+1}$$

    for all $n = 1, 2, 3, \cdots$. If $b_n = 1 + \dfrac{1}{a_n}$ for all $n = 1, 2, 3, \cdots$, find the largest integer m such that

$$\sum_{k=1}^{n} \frac{1}{\log_3 b_k} > \frac{m}{24}$$

    for all positive integer $n \geq 2$.

23. Find the largest real number p such that all three roots of the equation below are positive integers:

$$5x^3 - 5(p+1)x^2 + (71p - 1)x + 1 = 66p.$$

24. Let a, b, c, d be 4 distinct nonzero integers such that $a + b + c + d = 0$ and the number $M = (bc - ad)(ac - bd)(ab - cd)$ lies strictly between 96100 and 98000. Determine the value of M.

25. In the triangle ABC, $AB = 585$, $BC = 520$, $CA = 455$. Let P, Q be points on the side BC, and $R \neq A$ the intersection of the line AQ with the circumcircle ω of the triangle ABC. Suppose PR is parallel to AC and the circumcircle of the triangle PQR is tangent to ω at R. Find PQ.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2013
(Open Section, First round Solution)

1. Answer: 12877

   Solution. Let S be the required sum. By using method of difference,

$$\begin{aligned} S &= \frac{1}{2}\left(\frac{1}{1\times 2}-\frac{1}{2\times 3}+\frac{1}{2\times 3}-\frac{1}{3\times 4}+\cdots+\frac{1}{100\times 101}-\frac{1}{101\times 102}\right) \\ &= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{10302}\right) \\ &= \frac{2575}{10302}, \end{aligned}$$

   Hence $a+b=12877$. □

2. Answer: 1

   Solution. Let $y=\dfrac{1+\cos x}{\sin x+\cos x+2}$. When $\cos x+1=0$, $y=0$. Otherwise,

$$y=\frac{1}{1+\frac{1+\sin x}{1+\cos x}}.$$

   Let $u=\dfrac{1+\sin x}{1+\cos x}$. It is clear that $u\geq 0$, and so $y\leq 1$ where the equality holds when $u=0$.

   Thus the maximum value of y is 1 when $\sin x=-1$. □

3. Answer: 3

   Solution. We have $\tan\alpha+\tan\beta=3$ and $\tan\alpha\tan\beta=-3$. Hence

$$\tan(\alpha+\beta)=\frac{3}{1-(-3)}=\frac{3}{4}.$$

   Hence

$$\begin{aligned} &\left|\sin^2(\alpha+\beta)-3\sin(\alpha+\beta)\cos(\alpha+\beta)-3\cos^2(\alpha+\beta)\right| \\ =\ & \cos^2(\alpha+\beta)\left[\left(\frac{3}{4}\right)^2-3\left(\frac{3}{4}\right)-3\right] \\ =\ & \cos^2(\alpha+\beta)\times\left(-\frac{75}{16}\right), \end{aligned}$$

   and since

$$\tan^2(\alpha+\beta)=\frac{1-\cos^2(\alpha+\beta)}{\cos^2(\alpha+\beta)}=\frac{1}{\cos^2(\alpha+\beta)}-1=\frac{9}{16},$$

   we have $\cos^2(\alpha+\beta)=\dfrac{16}{25}$. Thus, we have

$$\left|\sin^2(\alpha+\beta)-3\sin(\alpha+\beta)\cos(\alpha+\beta)-3\cos^2(\alpha+\beta)\right|=|-3|=3.$$

□

4. Answer: 20

Solution. Let $a_n = a_1 + (n-1)d$. As

$$3a_8 = 5a_{13},$$

we have $3(a_1 + 7d) = 5(a_1 + 12d)$, and so $2a_1 + 39d = 0$. So $d < 0$ and

$$a_{20} + a_{21} = a_1 + 19d + a_1 + 20d = 0.$$

So $a_{20} > 0$ but $a_{21} < 0$, as $a_{21} = a_{20} + d$ and $a_{20} + a_{21} = 0$. Thus $a_1, a_2, a_3, a_4, \cdots$ is an decreasing sequence and

$$a_1 > a_2 > \cdots > a_{20} > 0 > a_{21} > \cdots.$$

Hence S_n has the maximum value when $n = 20$. $\square$

5. Answer: 81

Solution. Note that $\mathrm{f}(\mathrm{g}(x)) = \sin 2x = 2\sin x \cos x = \dfrac{4\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \cdot \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$. Hence

$$\mathrm{f}(\frac{\sqrt{2}}{2}) = \frac{2\sqrt{2}}{1+\frac{1}{2}} \cdot \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{4\sqrt{2}}{9}.$$

If $k\mathrm{f}(\frac{\sqrt{2}}{2}) = 36\sqrt{2}$, then $k = 81$. $\square$

6. Answer: 2

Solution. Note that $2t^2+t+5 = 2\left(t+\frac{1}{4}\right)^2 + \frac{39}{8} > 0$. Hence $\mathrm{g}(2t^2+t+5) < \mathrm{g}(t^2-3x+2)$ is true if and only if

$$2t^2 + t + 5 < t^2 - 3t + 2,$$

which is equivalent to $(t+3)(t+1) < 0$. Hence the range of t satisfying the given inequality is $-3 < t < -1$, which yields $a - b = (-1) - (-3) = 2$. $\square$

7. Answer: 240

Solution. By counting the number of squares of different types, we obtain

$$1\times 2 + 2\times 3 + 3\times 4 + 4\times 5 + 5\times 6 + 6\times 7 + 7\times 8 + 8\times 9 = \frac{8\times 9\times 10}{3} = 240.$$

$\square$

8. Answer: 135

Solution. Note that

$$(\sqrt{3a+12} + \sqrt{3b+12} + \sqrt{3c+12})^2 \le 3(3a+12+3b+12+3c+12)$$
$$= 9(a+b+c+12) = 9(2013+12) = 9\times 2025,$$

where the equality holds if $3a + 12 = 3b + 12 = 3c + 12$, i.e., $a = b = c = 671$. Thus the answer is $3 \times 45 = 135$.

$\square$

9. Answer: 75

Solution. Let

$$B = \sin^2 10^\circ + \sin^2 50^\circ - \cos 40^\circ \cos 80^\circ.$$

Then

$$A + B = 2 - \cos 40^\circ$$

and

$$\begin{aligned} A - B &= \cos 20^\circ + \cos 100^\circ + \cos 120^\circ = 2\cos 60^\circ \cos 40^\circ + \cos 120^\circ \\ &= \cos 40^\circ - \frac{1}{2}. \end{aligned}$$

Thus $2A = \frac{3}{2}$ and $100A = 75$. □

10. Answer: 6

Solution. Note that

$$\begin{aligned} 2 \sum_{1 \le i < j \le 2013} a_i a_j &= (a_1 + a_2 + \cdots + a_{2013})^2 - (a_1^2 + a_2^2 + \cdots + a_{2013}^2) \\ &= (a_1 + a_2 + \cdots + a_{2013})^2 - 2013. \end{aligned}$$

By the given condition, $a_1 + a_2 + \cdots + a_{2013}$ is an odd number between -2013 and 2013 inclusive.

Also note that the minimum positive integer of $x^2 - 2013$ for an integer x is $45^2 - 2013 = 12$ when $x = 45$ or -45. As an illustration, $x = 45$ can be achieved by taking $a_1 = a_2 = a_3 = \cdots = a_{45} = 1$ and the others $a_{46}, a_{47}, \cdots, a_{2013}$ to consist of equal number of 1's and -1's. Thus the least value is $\frac{12}{2} = 6$. □

11. Answer: 2

Solution. Letting $x = -y$, we get

$$-\frac{27\mathrm{f}(y)}{y} - y^2 \mathrm{f}\left(-\frac{1}{y}\right) = -2y^2. \tag{1}$$

Letting $x = \frac{1}{y}$, we get

$$27y\mathrm{f}\left(-\frac{1}{y}\right) - \frac{1}{y^2}\mathrm{f}(y) = -\frac{2}{y^2}. \tag{2}$$

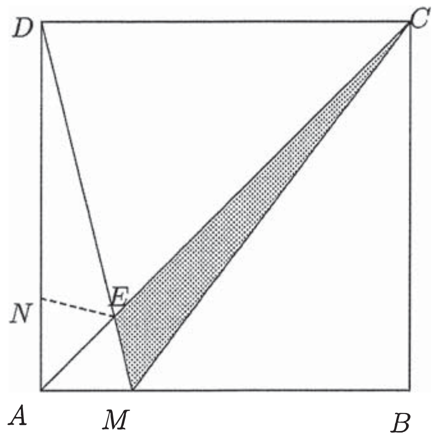
Then $27 \times (1) + y \times (2)$ gives

$$-\frac{729\mathrm{f}(y)}{y} - \frac{\mathrm{f}(y)}{y} = -54y^2 - \frac{2}{y}.$$

Solving for $\mathrm{f}(y)$, we have $\mathrm{f}(y) = \frac{1}{365}(27y^3 + 1)$. Thus $\mathrm{f}(3) = \frac{3^6+1}{365} = 2$. □

12. Answer: 5

Solution. Choose a point N on DA such that $NA = MA = x$.



It is clear that ΔNAE and ΔMAE are congruent by SAS test. Let S be the area of $\triangle NAE$. Then area of $\Delta DNE = \dfrac{20-x}{x}S$. It is also clear that areas of ΔDAE and ΔCEM are equal to 40cm^2. It follows that

$$\text{Area of } \Delta DAE = \frac{20-x}{x}S + S = \frac{20}{x}S,$$

so that $\dfrac{20}{x}S = 40\text{cm}^2$, that is, $S = 2x$.

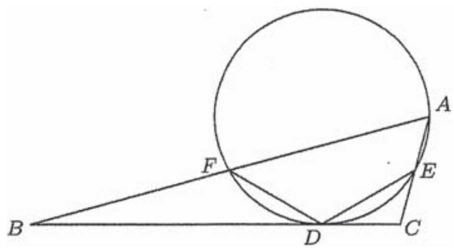
Area of $\Delta DAM = \dfrac{1}{2} \times x \times 20 = 10x$. On the other hand,

$$\begin{aligned}\text{Area of } \Delta DAM &= \text{Area of} \Delta DAE + \text{Area of} \Delta AEM \\ &= \frac{20}{x}S + S \\ &= \frac{20+x}{x}S = \frac{20+x}{x} \times 2x = 2(20+x).\end{aligned}$$

So $2(20+x) = 10x$, which means that $AM = x = 5\text{cm}$. □

13. Answer: 120

Solution.



Let $BC = a, CA = b$ and $AB = c$. Let $BD = a_1$ and $DC = a_2$. Using the power of B with respect to the circle, we have $a_1^2 = c^2/2$. Similarly, $a_2^2 = b^2/2$. Thus $b + c = \sqrt{2}(a_1 + a_2) =$

$\sqrt{2}a$, or $2a^2 = (b+c)^2$. Therefore,

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{2b^2+2c^2-(b+c)^2}{4bc} = \frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}-2\right) = \frac{1}{4}(4-2) = \frac{1}{2}.$$

Therefore, $\angle A = 60°$. Since A, F, D, E are concyclic, $\angle EDF = 120°$.

□

14. Answer: 150

Solution. Let r be the common ratio of this geometric sequence. Thus

$$S_n = a_1(1+r+r^2+\cdots+r^{n-1}).$$

Thus

$$10 = a_1(1+r+\cdots+r^9)$$

and

$$70 = a_1(1+r+\cdots+r^{29}).$$

As

$$1+r+\cdots+r^{29} = (1+r+\cdots+r^9)(1+r^{10}+r^{20}),$$

we have

$$1+r^{10}+r^{20} = 7.$$

So r^{10} is either 2 or -3. As $r^{10} > 0$, $r^{10} = 2$.

Hence

$$S_{40} = a_1(1+r+\cdots+r^{39}) = a_1(1+r+\cdots+r^9)(1+r^{10}+r^{20}+r^{30})$$
$$= 10 \times (1+2+2^2+2^3) = 150.$$

□

15. Answer: 106

Solution. Note that an integer is a multiple of 3 if and only if the sum of its digits is a multiple of 3. Also note that the sum of three integers a, b, c is a multiple of 3 if and only if either (i) a, b, c all have the same remainder when divided by 3, or (ii) a, b, c have the distinct remainders when divided by 3. Observe that the remainders of $0, 1, 2, 3, 4, 5, 6, 7$ when divided by 3 are $0, 1, 2, 0, 1, 2, 0, 1$ respectively.

For case (i), the only possible selections such that all the three numbers have the same remainder when divided by 3 are $\{0, 3, 6\}$ and $\{1, 4, 7\}$. With $\{0, 3, 6\}$, we have 4 possible numbers (note that a number does not begin with 0), and with $\{1, 4, 7\}$, there are 6 possible choices.

For case (ii), if the choice of the numbers does not include 0, then there are $2\times3\times2\times3! = 72$; if 0 is included, then there are $3 \times 2 \times 4$ choices.

Hence the total number of possible three-digit numbers is $72 + 24 + 10 = 106$.

□

16. Answer: 32

Solution. Note that $2012 = 2^2 \times 503$, and that 503 is a prime number. There are 1006 multiples of 2 less than or equal to 2012; there are 4 multiples of 503 less than or equal to 2012; there are 2 multiples of 1006 less than or equal to 2012. By the Principle of Inclusion and Exclusion, there are $1006 + 4 - 2 = 1008$ positive integers not more than 2012 which are not co-prime to 2012. Hence there are $2012 - 1008 = 1004$ positive integers less than 2012 which are co-prime with 2012. Thus, 2013 is the 1005^{th} number co-prime with 2012. Note also that the sum of the first n odd numbers equals n^2, and that $31^2 < 1005 < 32^2$, the number 2013 must be in the 326th group. Hence $k = 32$. □

17. Answer: 67

The total number of ways of dividing the seven numbers into two non-empty subsets is $\frac{2^7 - 2}{2} = 63$. Note that since $1 + 2 + 3 + \cdots + 7 = 28$, the sum of the numbers in each of the two groups is 14. Note also that the numbers $5, 6, 7$ cannot be in the same group since $5 + 6 + 7 = 18 > 14$. We consider three separate cases:

Case (i): Only 6 and 7 in the same group and 5 in the other group:

$\{2, 3, 4, 5\}, \{1, 6, 7\}$

Case (ii): Only 5 and 6 in the same group and 7 in the other group:

$\{1, 2, 5, 6\}, \{3, 4, 7\}$

$\{3, 5, 6\}, \{1, 2, 4, 7\}$

Case (iii): Only 5 and 7 in the same group and 6 in the other group:

$\{2, 5, 7\}, \{1, 3, 4, 6\}$

Hence there are 4 such possibilities. Thus the required probability is $\frac{4}{63}$, yielding that $p + q = 67$. □

18. Answer: 3

Solution. Let $u = \lfloor \log_{10} x \rfloor$ and $r = \log_{10} x - u$. So $0 \leq r < 1$. Thus

$$(u + r)^2 = u + 2.$$

Case 1: $r = 0$.

Then $u^2 = u + 2$ and so $u = 2$ or $u = -1$, corresponding to $x = 10^2 = 100$ and $x = 10^{-1} = 0.1$.

Case 2: $0 < r < 1$.

In this case, $u + 2$ is an integer which is not a complete square and

$$r = \sqrt{u + 2} - u.$$

As $r > 0$, we have $u \leq 2$. But $u + 2$ is not a complete square. So $u \leq 1$. As $u + 2 \geq 0$ and not a complete square, we have $u \geq 0$. Hence $u \in \{0, 1\}$.

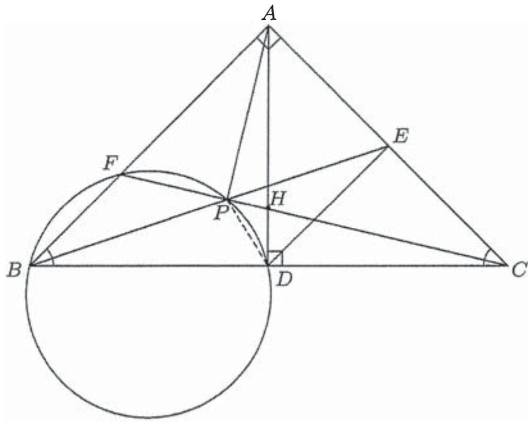
If $u = 0$, then $r = \sqrt{2} - 0 > 1$, not suitable.

If $u = 1$, then $r = \sqrt{3} - 1$. So $\log_{10} x = \sqrt{3}$ and $x = 10^{\sqrt{3}}$.

Hence the answer is 3. □

19. Answer: 1

Solution.



Join PD. Then $\angle DPC = \angle FBD = 45° = \angle DAC$ so that D, P, A, C are concyclic. Thus $\angle APC = \angle ADC = 90°$. It follows that $EA = EP = ED = EC$. Let $\angle PAH = \theta$. Then $\angle PCD = \theta$. Thus $\angle EPC = \angle ECP = 45° - \theta$ so that $\angle AEB = 90° - 2\theta$. That is $\angle ABE = 2\theta$. Thus $\tan 2\theta = AE/AB = 1/2$. From this, we get $\tan\theta = \sqrt{5} - 2$. Therefore, $PH = AP\tan\theta = (\sqrt{5}+2)(\sqrt{5}-2) = 1$. □

20. Answer: 1611

Solution. Note that $n^4+5n^2+9 = n^4-1+5n^2+10 = (n-1)(n+1)(n^2+1)+5(n^2+2)$.

If $n \equiv 1$ or $4 (\text{mod } 5)$, then 5 divides $n-1$ or $n+1$.

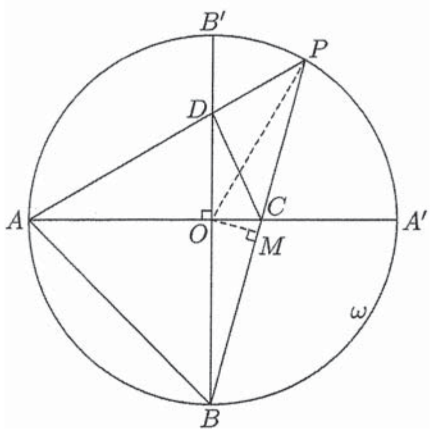
If $n \equiv 2$ or $3 (\text{mod } 5)$, then 5 divides n^2+1.

If $n \equiv 0 (\text{mod } 5)$, then 5 does not divide $(n-1)n(n^2+1)$ but divides $5(n^2+2)$, hence does not divide n^4+5n^2+9.

Thus, there are $2010 \div 5 = 402$ multiples of 5 from 1 to 2013. The number of integers thus required is $2013 - 402 = 1611$. □

21. Answer: 10

Solution. Since AC intersects BD at right angle, the area of the convex quadrilateral $ABCD$ is $\frac{1}{2}AC \cdot BD$. Let M be the midpoint of PB. As $\angle CAB = \angle ABD = 45°$, and $\angle BCA = \angle BOM = \angle DAB$, we have $\triangle ABC$ is similar to $\triangle BDA$. Thus $AB/BD = AC/BA$. From this, we have $(ABCD) = \frac{1}{2}AC \cdot BD = \frac{1}{2}AB^2 = OA^2$ so that $OA = 10$.



□

22. Answer: 13

Solution. Given that $a_n = 2a_na_{n+1} + 3a_{n+1}$ we obtain $a_{n+1} = \dfrac{a_n}{2a_n+3}$. Thus we have $\dfrac{1}{a_{n+1}} = 2 + \dfrac{3}{a_n}$. We thus have $\dfrac{1}{a_{n+1}} + 1 = 3\left(1 + \dfrac{1}{a_n}\right)$ for all $n = 1, 2, 3, \cdots$. Letting $b_n = 1 + \dfrac{1}{a_n}$, it is clear that the sequence $\{b_n\}$ follows a geometric progression with first term $b_1 = 1 + \dfrac{1}{a_1} = 3$, and common ratio 3. Thus, for $n = 1, 2, 3, \cdots$, $b_n = 1 + \dfrac{1}{a_n} = 3^n$ for $n = 1, 2, 3, \cdots$.

Let $f(n) = \displaystyle\sum_{k=1}^{n} \frac{1}{n + \log_3 b_k} = \sum_{k=1}^{n} \frac{1}{n+k} > \frac{m}{24}$, $n = 2, 3, 4, \cdots$. It is clear that $f(n)$ is an increasing function since

$$f(n+1) - f(n) = \frac{1}{n+1} > 0.$$

Thus $f(n)$ is a strictly increasing sequence in n. Thus the minimum value of $f(n)$ occurs when $n = 2$.

$$f(2) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{m}{24},$$

forcing $m < 14$. Thus the largest value of integer m is 13. □

23. Answer: 76

Solution. Observe that $x = 1$ is always a root of the equation

$$5x^3 - 5(p+1)x^2 + (71p - 1)x + 1 = 66p.$$

Thus this equation has all roots positive integers if and only if the two roots of the equation below are positive integers:

$$5x^2 - 5px + 66p - 1 = 0.$$

Let u, v be the two roots with $u \le v$. Then

$$u + v = p, \quad uv = (66p - 1)/5,$$

implying that

$$5uv = 66(u+v) - 1.$$

By this expression, we know that u, v are not divisible by any one of $2, 3, 11$. We also have $5uv > 66(u+v)$, implying that

$$\frac{2}{u} \ge \frac{1}{u} + \frac{1}{v} > \frac{5}{66},$$

and so $u \le 26$. As

$$v = \frac{66u - 1}{5u - 66} > 0,$$

we have $5u - 66 > 0$ and so $u \ge 14$. Since u is not a multiple of any one of $2, 3, 11$, we have

$$u \in \{17, 19, 23, 25\}.$$

As $v = \frac{66u-1}{5u-66}$, only when $u = 17$, $v = 59$ is an integer.

Thus, only when $p = u + v = 17 + 69 = 76$, the equation

$$5x^3 - 5(p+1)x^2 + (71p - 1)x + 1 = 66p$$

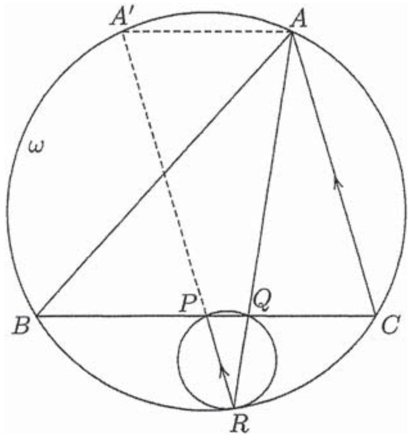
has all three roots being positive integers. $\square$

24. Answer: 97344

Solution. First we show that M must be a square. Let $d = -a - b - c$. Then $bc - ad = bc + a(a + b + c) = (a + c)(a + b)$, $ac - bd = ac + b(a + b + c) = (b + c)(b + a)$, and $ab - cd = ab + c(a + b + c) = (c + a)(c + b)$. Therefore $M = (a + b)^2(b + c)^2(c + a)^2$. Note that $(a+b)(b+c)(c+a)$ cannot be an odd integer since two of the 3 numbers a, b, c must be of the same parity. The only squares in $(96100, 98000)$ are $311^2, 312^2, 313^2$. Since 311 and 313 are odd, the only value of M is $312^2 = 97344$. When $a = 18, b = -5, c = 6, d = -19$, it gives $M = 97344$. $\square$

25. Answer: 64

Solution.



First by cosine rule, $\cos C = 2/7$. Reflect A about the perpendicular bisector of BC to get the point A' on ω. Then $AA'BC$ is an isosceles trapezoid with $A'A$ parallel to BC. Thus $A'A = BC - 2AC\cos C = 520 - 2 \times 455 \times 2/7 = 260$. Consider the homothety h centred at R mapping the circumcircle of PQR to ω. We have $h(Q) = A$, and $h(P) = A'$ because PQ is parallel to $A'A$. Thus A', P, R are collinear and $AA'PC$ is a parallelogram. Hence $PC = AA' = 260$, and P is the midpoint of BC. Also $PA' = CA = 455$. As $PA' \times PR = BP \times PC$, we have $455 \times PR = 260^2$ giving $PR = 1040/7$. Since the triangles PQR and $A'AR$ are similar, we have $PQ/A'A = RP/RA'$. Therefore, $PQ = 260 \times (1040/7)/(455 + 1040/7) = 64$. $\square$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Open Section, Round 2)

Saturday, 6 July 2013 **0900-1300**

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1. Let $a_1, a_2, \ldots$ be a sequence of integers defined recursively by $a_1 = 2013$ and for $n \geq 1$, a_{n+1} is the sum of the 2013^{th} power of the digits of a_n. Do there exist distinct positive integers i, j such that $a_i = a_j$?

2. Let ABC be an acute-angled triangle and let D, E and F be the midpoints of BC, CA and AB respectively. Construct a circle, centred at the orthocentre of triangle ABC, such that triangle ABC lies in the interior of the circle. Extend EF to intersect the circle at P, FD to intersect the circle at Q and DE to intersect the circle at R. Show that $AP = BQ = CR$.

3. Let N be a positive integer. Prove that there exists a positive integer n such that $n^{2013} - n^{20} + n^{13} - 2013$ has at least N distinct prime factors.

4. Let F be a finite nonempty set of integers and let n be a positive integer. Suppose that

- Any $x \in F$ may be written as $x = y + z$ for some $y, z \in F$;
- If $1 \leq k \leq n$ and $x_1, \ldots, x_k \in F$, then $x_1 + \cdots + x_k \neq 0$.

Show that F has at least $2n + 2$ distinct elements.

5. Let ABC be a triangle with integral side lengths such that $\angle A = 3\angle B$. Find the minimum value of its perimeter.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

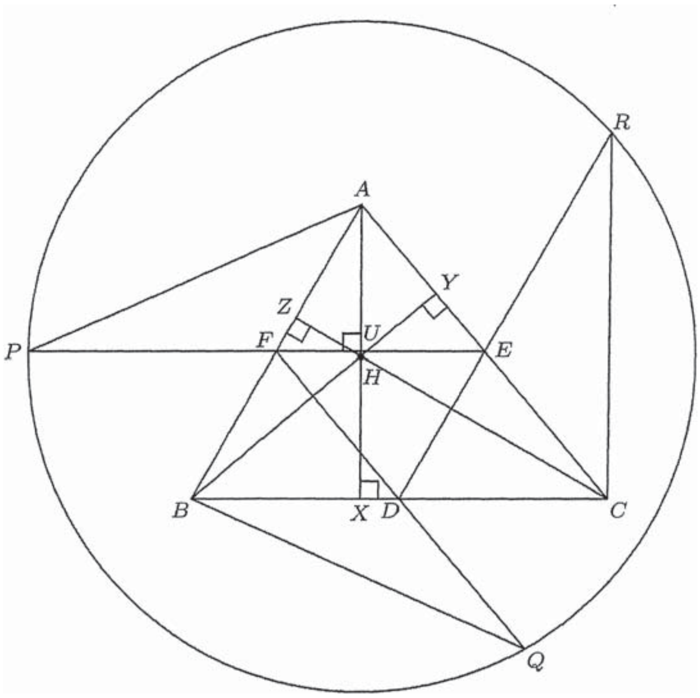
(Open Section, Round 2 solutions)

1. The answer is yes. For any positive integer n, let $f(n)$ be the sum of 2013^{th} power of the digits of n. Let $S = \{1, 2, \ldots, 10^{2017} - 1\}$, and $n = \overline{a_1 a_2 \ldots a_{2017}} \in S$. Then

$$f(n) = \sum a_i^{2013} \leq 2017 \cdot 9^{2013} < 10^4 \cdot 10^{2013} = 10^{2017} \in S.$$

Since $a_i = f^{(i)}(2013) \in S$, there exist distinct positive integers i, j such that $a_i = a_j$.

2. Let the radius of the circle be r. Let X, Y and Z be the feet of the altitudes from A, B and C respectively. Let PE intersect the altitude from A at U. We have $AP^2 = AU^2 + PU^2 = AU^2 + r^2 - UH^2 = r^2 + (AU+UH)\cdot(AU-UH) = r^2 + AH\cdot(AU-UH) = r^2 + AH\cdot(UX-UH) = r^2 + AH\cdot HX$. Similarly, $BQ = r^2 + BH\cdot HY$, and $CR = r^2 + CH\cdot HZ$. Since $AH\cdot HX = BH\cdot HY = CH\cdot HZ$, we have $AP = BQ = CR$.



3. The result is true for any nonconstant polynomial $f(n) = a_m n^m + a_{m-1}n^{m-1} + \cdots + a_0$ with integer coefficients. We may assume that $a_m > 0$. Thus there exists a positive integer n_0 such that $f(n)$ is positive and increasing on (n_0, ∞).

It suffices to show that if for some $n_1 > n_0$, $f(n_1) = p_1^{r_1} \cdots p_k^{r_k}$ has exactly k distinct prime factors, then for some $n_2 > n_1$, $f(n_2)$ has more than k prime factors. Given such an n_1, let $n_2 = n_1 + p_1^{r_1+1} \cdots p_k^{r_k+1}$. Then

$$f(n_2) \equiv p_1^{r_1} \cdots p_k^{r_k} \pmod{p_1^{r_1+1} \cdots p_k^{r_k+1}}.$$

Hence, for each j, $1 \leq j \leq k$, we have that $p_j^{r_j}$ divides $f(n_2)$ but $p_j^{r_j+1}$ does not divide $f(n_2)$. As $f(n_2) > f(n_1) = p_1^{r_1} \cdots p_k^{r_k}$, it follows that $f(n_2)$ must have at least $k+1$ prime factors.

4. Because of the second condition above, $0 \notin F$. If F contains only positive elements, let x be the smallest element in F. But then $x = y + z$, and $y, z > 0$ imply that $y, z < x$, a contradiction. Hence F contains negative elements. A similar argument shows that F contains positive elements.

Pick any positive element of F and label it as x_1. Assume that positive elements of F, $x_1, \ldots, x_k$, have been chosen. We can write $x_k = y + z$, where $y, z \in F$. We may assume that $y > 0$. Label y as x_{k+1}. Carry on in this manner to choose positive elements $x_1, x_2, \ldots$ of F, not necessarily distinct. Since F is a finite set, there exist positive integers $i < j$ such that $x_i, \ldots, x_{j-1}$ are distinct and $x_j = x_i$. There are $z_i, \ldots z_{j-1} \in F$ such that

$$\begin{aligned} x_i &= x_{i+1} + z_i \\ x_{i+1} &= x_{i+2} + z_{i+1} \\ \vdots \quad & \quad \vdots \\ x_{j-1} &= x_j + z_{j-1}. \end{aligned}$$

Since $x_i = x_j$, we see that $z_i + z_{i+1} + \cdots + z_{j-1} = 0$. By the assumption, $j - i > n$. Since the elements $x_i, \ldots, x_{j-1}$ are distinct, F contains at least $j - i \geq n + 1$ positive elements. Similarly, F contains at least $n + 1$ negative elements. The result follows.

5. Let the sides be a, b, c. From the sine rule, we have

$$\begin{aligned} \frac{a}{b} &= \frac{\sin 3B}{\sin B} = 4\cos^2 B - 1 \\ \frac{c}{b} &= \frac{\sin C}{\sin B} = \frac{\sin 4B}{\sin B} = 8\cos^3 B - 4\cos B \end{aligned}$$

Thus

$$2\cos B = \frac{a^2 + c^2 - b^2}{ac} \in \mathbb{Q}.$$

Hence there exist coprime positive integers p, q such that $2\cos B = \frac{p}{q}$. Hence

$$\frac{a}{b} = \frac{p^2}{q^2} - 1 \quad \Leftrightarrow \quad \frac{a}{p^2 - q^2} = \frac{b}{q^2};$$
$$\frac{c}{b} = \frac{p^3}{q^3} - \frac{2p}{q} \quad \Leftrightarrow \quad \frac{c}{p^3 - 2pq^2} = \frac{b}{q^3}.$$

Thus

$$\frac{a}{(p^2 - q^2)q} = \frac{b}{q^3} = \frac{c}{p^3 - 2pq^2} = \frac{e}{f}, \qquad \gcd(e, f) = 1.$$

Since perimeter is minimum, $\gcd(a, b, c) = 1$. From $\gcd(e, f) = 1$, we have $f \mid q^3$ and $f \mid p^3 - 2pq^2$. We'll prove that $f = 1$.

If $f > 1$, then it has a prime divisor $f' > 1$ such that $f' \mid q^3$ and $f' \mid p^3 - 2pq^2$. Thus $f' \mid q$ and $f' \mid p$, contradicting $\gcd(p, q) = 1$. Thus $f = 1$. From $\gcd(a, b, c) = 1$, we conclude that $e = 1$. Thus

$$a = (p^2 - q^2)q, \quad b = q^3, \quad c = p^3 - 2pq^2.$$

From $0^\circ < \angle A + \angle B = 4\angle B < 180^\circ$, we get $0^\circ < \angle B < 45^\circ$ and hence $\sqrt{2} < 2\cos B < 2$ implying that $\sqrt{q} < p < 2q$. The smallest positive integers satisfying this inequality is $p = 3, q = 2$. Since $a + b + c = p^2 q + p(p^2 - 2q^2)$ and $p^2 - 2q^2 = 1$, we see that the minimum perimeter is achieved when $p = 3, q = 2$ and the value is 21.